DEFINITION: Let R be a ring, M an R-module, and W a multiplicatively closed subset. The **localization** $W^{-1}M$ is the $W^{-1}R$-module[1] with

- elements equivalence classes of $(m, w) \in M \times W$, with the class of (m, w) denoted as $\dfrac{m}{w}$.
- with equivalence relation $\dfrac{m}{u} = \dfrac{n}{v}$ if there is some $w \in W$ such that $w(vm - un) = 0$,
- addition given by $\dfrac{m}{u} + \dfrac{n}{v} = \dfrac{vm + un}{uv}$, and
- action given by $\dfrac{r}{u}\dfrac{m}{v} = \dfrac{rm}{uv}$.

If $\alpha : M \to N$ is a homomorphism of R-modules, then the $W^{-1}R$-module homomorphism $W^{-1}\alpha : W^{-1}M \to W^{-1}N$ is defined by $W^{-1}\alpha(\frac{m}{w}) = \frac{\alpha(m)}{w}$.

DEFINITION: Let R be a ring and M a module.

- If $f \in R$, then $M_f := \{1, f, f^2, \dots\}^{-1}M$.
- If $\mathfrak{p} \subseteq R$ is a prime ideal then $M_{\mathfrak{p}} := (R \smallsetminus \mathfrak{p})^{-1}M$.

PROPOSITION: Let R be a ring, W a multiplicatively closed set, and $N \subseteq M$ be modules. Then

- $W^{-1}N$ is a submodule of $W^{-1}M$, and
- $W^{-1}(M/N) \cong \dfrac{W^{-1}M}{W^{-1}N}$.

COROLLARY: Let R be a ring, I an ideal, and W a multiplicatively closed subset. Then the map $R \to W^{-1}(R/I)$ induces an order preserving bijection

$$\mathrm{Spec}(W^{-1}(R/I)) \xrightarrow{\sim} \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq I \text{ and } \mathfrak{p} \cap W = \varnothing\}.$$

(1) Let M be an R-module and W be a multiplicatively closed set.

(a) What is the natural map from $M \to W^{-1}M$?

(b) If S is a generating set for M, explain why $\frac{S}{1} = \{\frac{s}{1} \mid s \in S\}$ is a generating set for $W^{-1}M$.

(c) Let $m \in M$. Show that $\frac{m}{u}$ is zero in $W^{-1}M$ if and only if there is some $w \in W$ such that $wm = 0$ in M.

(d) Let $m_1, \dots, m_t \in M$ be a finite set of elements. Show that $\frac{m_1}{u_1}, \dots, \frac{m_t}{u_t} \in W^{-1}M$ are all zero if and only if there is some $w \in W$ that such that $wm_i = 0$ in M for all i.

(e) Let M be a finitely generated module. Show that $W^{-1}M = 0$ if and only if $M_w = 0$ for some $w \in W$.

(f) Let $m \in M$ and $\mathfrak{p}$ be a prime ideal. Show that $\frac{m}{1} \neq 0$ in $M_{\mathfrak{p}}$ if and only if $\mathfrak{p} \supseteq \mathrm{ann}_R(m)$.

(a) $m \mapsto \frac{m}{1}$

(b) We can write $\frac{m}{w} = \frac{\sum_i r_i m_i}{w} = \sum_i \frac{r_i}{w}\frac{m_i}{1}$.

(c) $\frac{m}{u} = \frac{0}{1}$ iff $\exists w$ such that $0 = w(1m - 0u) = wm$.

(d) The "if" is clear; for the only if, we have $w_1 m_1 = \cdots w_t m_t = 0$ so we can take $w = w_1 \cdots w_t$.

[1] If $0 \in W$, then $W^{-1}R = 0$ is not a ring.

(e) Take a finite generating set for M. Then $W^{-1}M = 0$ iff each generator maps to 0 iff there is a w that kills each m_i iff the corresponding $M_w = 0$.

(f) $\frac{m}{1} = 0$ if and only if there is some $w \notin \mathfrak{p}$ with $wm = 0$, which happens if and only if $\mathfrak{p} \not\supseteq \mathrm{ann}_R(m)$.

(2) Prove the Proposition.

For the first part, we need to show that a nonzero element in $W^{-1}N$ is nonzero in $W^{-1}M$. If $\frac{n}{u} \neq 0$, in $W^{-1}M$ then there is some $w \in W$ such that $wn = 0$, which is the same as the condition to be zero in $W^{-1}N$.

For the second part, consider the map from $W^{-1}M$ to $W^{-1}(M/N)$ given by $\frac{m}{u} \mapsto \overline{m}u$. Clearly, $W^{-1}N$ is contained in the kernel. An element is in the kernel if and only if there is some $w \in W$ such that $w\overline{m} = 0$ in M/N, which means $wm \in N$. Then $\frac{m}{u} = \frac{wm}{wu} \in W^{-1}N$.

(3) Corollary.

(a) Rewrite the Corollary in the special case $W = R \setminus \mathfrak{p}$ for some prime $\mathfrak{p}$.

(b) Use the Proposition[2] to justify the Corollary.

(a) There is a bijection between $\mathrm{Spec}((R/I)_{\mathfrak{p}})$ and primes of R containing I but also contained in $\mathfrak{p}$.

(b) We have $W^{-1}(R/I) \cong W^{-1}R/W^{-1}I$. Fromt he Proposition, this is an isomorphism of R-modules, but it is easy to see that the map is in fact a ring isomorphism. The primes in $W^{-1}R$ are of the form $W^{-1}\mathfrak{p}$ for $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\mathfrak{p} \cap W = \varnothing$. By the lattice isomorphism theorem, the primes in $W^{-1}R/W^{-1}I$ correspond to primes $W^{-1}\mathfrak{p}$ that contain $W^{-1}I$. But if $\mathfrak{p} \supseteq I$ then $W^{-1}\mathfrak{p} \supseteq W^{-1}I$, and if $W^{-1}\mathfrak{p} \supseteq W^{-1}I$, then since $W^{-1}\mathfrak{p} \cap R = \mathfrak{p}$ (from definition of prime) $I \subseteq W^{-1}I \cap R \subseteq W^{-1}\mathfrak{p} \cap R = \mathfrak{p}$. Thus, there is a bijection between primes containing I and not intersecting W with primes of $W^{-1}(R/I)$.

(4) Invariance of base: Let $\phi : R \to S$ be a ring homomorphism, and $V \subseteq R$ and $W \subseteq S$ be multiplicatively closed sets such that $\phi(V) = W$. Show that for any S-module M, $V^{-1}M \cong W^{-1}M$.

(5) I'm already local!

(a) Suppose that the action of each $w \in W$ on M is invertible: for every $w \in W$ the map $m \mapsto mw$ is bijective. Show that $M \cong W^{-1}M$ via the natural map.

(b) Let R be a ring, $\mathfrak{m}$ a maximal ideal (so $R/\mathfrak{m}$ is a field), and M a module such that $\mathfrak{m}M = 0$. Show that $M \cong M_{\mathfrak{m}}$ by the natural map.

(c) More generally, show that[3] if for every $m \in M$ there is some n such that $\mathfrak{m}^n m = 0$, then $M \cong M_{\mathfrak{m}}$.

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[2]Hint: You may want to show that, for $W \cap \mathfrak{p} = \varnothing$, $I \subseteq \mathfrak{p}$ if and only if $W^{-1}I \subseteq W^{-1}\mathfrak{p}$. For this, it may help to observe that $W^{-1}\mathfrak{p} \cap R = \mathfrak{p}$. You can also use that the isomorphism from the Proposition is a ring isomorphism when R is a ring and I is an ideal.

[3]Hint: Note that $R/\mathfrak{m}^n$ is local with maximal ideal (the image of) $\mathfrak{m}$.

(a) The map is injective, since $wm = 0$ implies $m = 0$, and surjective since $\frac{m}{w} = \frac{m'w}{w} = \frac{m'}{1}$ for some m'.

(b) Let $u \in R \setminus \mathfrak{m}$. Then since $R/\mathfrak{m}$ is a field, there is some $v \in R$ such that $uv \equiv 1 \mod \mathfrak{m}$. Then for any $m \in M$, we have $uvm = (1+a)m = m$ for some $a \in \mathfrak{m}$. In particular the action of v is the inverse of u.

(c) Because $R/\mathfrak{m}^n$ is local with maximal ideal $\mathfrak{m}$, every element not in $\mathfrak{m}$ in this ring is a unit. Thus, given $u \in R \setminus \mathfrak{m}$, there is some $v \in R$ such that $uv \equiv 1 \mod \mathfrak{m}^n$. This shows that the action of u on M is bijective and the first part applies.

(6) Prove the following:

LEMMA: Let R be a ring, W a multiplicatively closed set. Let M be a finitely presented[4] R-module, and N an arbitrary R-module. Then for any homomorphism of $W^{-1}R$-modules $\beta : W^{-1}M \to W^{-1}N$, there is some $w \in W$ and some R-module homomorphism $\alpha : M \to N$ such that $\beta = \frac{1}{w}W^{-1}\alpha$.

(a) Given β, show that there exists some $u \in W$ such that for every $m \in M$, $\frac{u}{1}\beta(\frac{M}{1}) \subseteq \frac{N}{1}$.

(b) Let $m_1, \dots, m_a$ be a (finite) set of generators for M, and $A = [r_{ij}]$ be a corresponding (finite) matrix of relations. Let $n_1, \dots, n_a$ be an a-tuple of elements of N. Justify: There exists an R-module homomorphism $\alpha : M \to N$ such that $\alpha(m_i) = n_i$ if and only if $[n_1, \cdots, n_a]A = 0$.

(c) Complete the proof.

(a) Let $m_1, \dots, m_a$ be a (finite) set of generators for M. We have $\beta(\frac{m_i}{1}) = \frac{t_i}{w_i}$ for some $t_i \in N$ and $w_i \in W$. Take $u = w_1 \cdots w_a$.

(b) For α to be well-defined means that relations map to zero; it suffices to show that any defining relation maps to zero, and the condition above just says this.

(c) In the notation of the above, let $\frac{n_i'}{u} = \beta(m_i)$. Note that

$$[\frac{n_1'}{u}, \cdots, \frac{n_a'}{u}]A = [\beta m_1, \cdots, \beta m_a]A = \beta([m_1, \dots, m_a]A) = 0 \qquad \text{in } W^{-1}N.$$

But this just means that there is some $v \in W$ such that v kills each entry of $[\frac{n_1'}{u}, \cdots, \frac{n_a'}{u}]A$. But then

$$[vn_1', \cdots, vn_a']A = (uv)[\frac{n_1'}{u}, \cdots, \frac{n_a'}{u}]A = 0.$$

This means that the map α given by $\alpha(m_i) = vn_i'$ is well defined, and $\beta = \frac{1}{uv}W^{-1}\alpha$ since it is true for each generator m_i.

[4]This means that M admits a finite generating set for which the module of relations is also finitely generated.