DEFINITION: Let R be a ring, M an R-module, and W a multiplicatively closed subset. The **localization** $W^{-1}M$ is the $W^{-1}R$-module[1] with

- elements equivalence classes of $(m,w) \in M \times W$, with the class of (m,w) denoted as $\dfrac{m}{w}$.
- with equivalence relation $\dfrac{m}{u} = \dfrac{n}{v}$ if there is some $w \in W$ such that $w(vm - un) = 0$,
- addition given by $\dfrac{m}{u} + \dfrac{n}{v} = \dfrac{vm + un}{uv}$, and
- action given by $\dfrac{r}{u}\dfrac{m}{v} = \dfrac{rm}{uv}$.

If $\alpha : M \to N$ is a homomorphism of R-modules, then the $W^{-1}R$-module homomorphism $W^{-1}\alpha : W^{-1}M \to W^{-1}N$ is defined by $W^{-1}\alpha(\frac{m}{w}) = \frac{\alpha(m)}{w}$.

DEFINITION: Let R be a ring and M a module.

- If $f \in R$, then $M_f := \{1, f, f^2, \dots\}^{-1}M$.
- If $\mathfrak{p} \subseteq R$ is a prime ideal then $M_{\mathfrak{p}} := (R \smallsetminus \mathfrak{p})^{-1}M$.

PROPOSITION: Let R be a ring, W a multiplicatively closed set, and $N \subseteq M$ be modules. Then

- $W^{-1}N$ is a submodule of $W^{-1}M$, and
- $W^{-1}(M/N) \cong \dfrac{W^{-1}M}{W^{-1}N}$.

COROLLARY: Let R be a ring, I an ideal, and W a multiplicatively closed subset. Then the map $R \to W^{-1}(R/I)$ induces an order preserving bijection

$$\operatorname{Spec}(W^{-1}(R/I)) \xrightarrow{\sim} \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} \supseteq I \text{ and } \mathfrak{p} \cap W = \varnothing\}.$$

(1) Let M be an R-module and W be a multiplicatively closed set.
  - **(a)** What is the natural map from $M \to W^{-1}M$?
  - **(b)** If S is a generating set for M, explain why $\frac{S}{1} = \{\frac{s}{1} \mid s \in S\}$ is a generating set for $W^{-1}M$.
  - **(c)** Let $m \in M$. Show that $\frac{m}{u}$ is zero in $W^{-1}M$ if and only if there is some $w \in W$ such that $wm = 0$ in M.
  - **(d)** Let $m_1, \dots, m_t \in M$ be a finite set of elements. Show that $\frac{m_1}{u_1}, \dots, \frac{m_t}{u_t} \in W^{-1}M$ are all zero if and only if there is some $w \in W$ that such that $wm_i = 0$ in M for all i.
  - **(e)** Let M be a finitely generated module. Show that $W^{-1}M = 0$ if and only if $M_w = 0$ for some $w \in W$.
  - **(f)** Let $m \in M$ and $\mathfrak{p}$ be a prime ideal. Show that $\frac{m}{1} \neq 0$ in $M_{\mathfrak{p}}$ if and only if $\mathfrak{p} \supseteq \operatorname{ann}_R(m)$.

(2) Prove the Proposition.

(3) Corollary.
  - **(a)** Rewrite the Corollary in the special case $W = R \smallsetminus \mathfrak{p}$ for some prime $\mathfrak{p}$.
  - **(b)** Use the Proposition[2] to justify the Corollary.

[1] If $0 \in W$, then $W^{-1}R = 0$ is not a ring.

[2] Hint: You may want to show that, for $W \cap \mathfrak{p} = \varnothing$, $I \subseteq \mathfrak{p}$ if and only if $W^{-1}I \subseteq W^{-1}\mathfrak{p}$. For this, it may help to observe that $W^{-1}\mathfrak{p} \cap R = \mathfrak{p}$. You can also use that the isomorphism from the Proposition is a ring isomorphism when R is a ring and I is an ideal.

(4) Invariance of base: Let $\phi : R \to S$ be a ring homomorphism, and $V \subseteq R$ and $W \subseteq S$ be multiplicatively closed sets such that $\phi(V) = W$. Show that for any S-module M, $V^{-1}M \cong W^{-1}M$.

(5) I'm already local!
  (a) Suppose that the action of each $w \in W$ on M is invertible: for every $w \in W$ the map $m \mapsto mw$ is bijective. Show that $M \cong W^{-1}M$ via the natural map.
  (b) Let R be a ring, $\mathfrak{m}$ a maximal ideal (so $R/\mathfrak{m}$ is a field), and M a module such that $\mathfrak{m}M = 0$. Show that $M \cong M_{\mathfrak{m}}$ by the natural map.
  (c) More generally, show that[3] if for every $m \in M$ there is some n such that $\mathfrak{m}^n m = 0$, then $M \cong M_{\mathfrak{m}}$.

(6) Prove the following:
LEMMA: Let R be a ring, W a multiplicatively closed set. Let M be a finitely presented[4] R-module, and N an arbitrary R-module. Then for any homomorphism of $W^{-1}R$-modules $\beta : W^{-1}M \to W^{-1}N$, there is some $w \in W$ and some R-module homomorphism $\alpha : M \to N$ such that $\beta = \frac{1}{w}W^{-1}\alpha$.
  (a) Given β, show that there exists some $u \in W$ such that for every $m \in M$, $\frac{u}{1}\beta(\frac{M}{1}) \subseteq \frac{N}{1}$.
  (b) Let $m_1, \ldots, m_a$ be a (finite) set of generators for M, and $A = [r_{ij}]$ be a corresponding (finite) matrix of relations. Let $n_1, \ldots, n_a$ be an a-tuple of elements of N. Justify: There exists an R-module homomorphism $\alpha : M \to N$ such that $\alpha(m_i) = n_i$ if and only if $[n_1, \cdots, n_a]A = 0$.
  (c) Complete the proof.

[3]Hint: Note that $R/\mathfrak{m}^n$ is local with maximal ideal (the image of) $\mathfrak{m}$.
[4]This means that M admits a finite generating set for which the module of relations is also finitely generated.