DEFINITION: Let R be a ring and W a multiplicatively closed subset with $0 \notin W$. The **localization** $W^{-1}R$ is the ring with

- elements equivalence classes of $(r, w) \in R \times W$, with the class of (r, w) denoted as $\dfrac{r}{w}$.
- with equivalence relation $\dfrac{s}{u} = \dfrac{t}{v}$ if there is some $w \in W$ such that $w(sv - tu) = 0$,
- addition given by $\dfrac{s}{u} + \dfrac{t}{v} = \dfrac{sv + tu}{uv}$, and
- multiplication given by $\dfrac{s}{u}\dfrac{t}{v} = \dfrac{st}{uv}$.

(If $0 \in W$, then $W^{-1}R := 0$, which by our convention is not a ring.)

DEFINITION: Let R be a ring.

- If $f \in R$ is nonnilpotent[1], then $R_f := \{1, f, f^2, \dots\}^{-1}R$.
- If $\mathfrak{p} \subseteq R$ is a prime ideal then $R_{\mathfrak{p}} := (R \smallsetminus \mathfrak{p})^{-1}R$.
- The **total quotient ring** of R is $\mathrm{Frac}(R) := \{w \in R \mid w \text{ is a nonzerodivisor}\}^{-1}R$.

For a ring R, multiplicative set $W \not\ni 0$, and an ideal I, we define

$$W^{-1}I := \left\{ \frac{a}{w} \in W^{-1}R \mid a \in I \right\}.$$

THEOREM: Let R be a ring and W be a multiplicatively closed subset. Then the map induced on Spec corresponding to the natural map $R \to W^{-1}R$ yields a homeomorphism into its image:

$$\mathrm{Spec}(W^{-1}R) \cong \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \cap W = \varnothing\}.$$

LEMMA: Let R be a ring and W be a multiplicatively closed subset.

(1) For any ideal $I \subseteq R$, $W^{-1}I = I(W^{-1}R)$.
(2) For any ideal $I \subseteq R$, $W^{-1}I \cap R = \{r \in R \mid \exists w \in W : wr \in I\}$.
(3) For any ideal $J \subseteq W^{-1}R$, $W^{-1}(J \cap R) = J$.
(4) For any prime ideal $\mathfrak{p} \subseteq R$ with[2] $\mathfrak{p} \cap W = \varnothing$, $W^{-1}\mathfrak{p}$ is prime.

(1) Computing localizations

- **(a)** What is the natural ring homomorphism $R \to W^{-1}R$?
- **(b)** Show that the kernel of $R \to W^{-1}R$ is ${}^{W}0 := \{r \in R \mid \exists w \in W : wr = 0\}$.
- **(c)** If every element of W is a nonzerodivisor, explain why the equivalence relation on $W^{-1}R$ simplifies to $\frac{s}{u} = \frac{t}{v}$ if and only if $sv = tu$.
- **(d)** If R is a domain, explain why $\mathrm{Frac}(R)$ is the usual fraction field of R.
- **(e)** If R is a domain, explain why $W^{-1}R$ is a subring of the fraction field of R. Which subring?
- **(f)** Let $\overline{R} = R/{}^{W}0$ and $\overline{W}$ be the image of W in $\overline{R}$. Show that $W^{-1}R \cong \overline{W}^{-1}\overline{R}$.

[1] If f is nilpotent, $0 \in \{1, f, f^2, \dots\}$ so $R_f = 0$.
[2] If $W \cap \mathfrak{p} \ni a$, then $W^{-1}\mathfrak{p} \ni \frac{a}{a} = \frac{1}{1}$, so $W^{-1}\mathfrak{p} = W^{-1}R$ is the improper ideal!

(a) $r \mapsto \frac{r}{1}$.
(b) $\frac{r}{1} = \frac{0}{1}$ if and only if $\exists w \in W : rw = w(1r - 0) = 0$.
(c) $w(sv - tu) = 0$ and w a nonzerdivisor implies $sv - tu = 0$; i.e., $sv = tu$.
(d) In light of the above, it's just the definition.
(e) The equivalence relation on the fractions is the same as that in the fraction field, so the map is injective; the operations are definitely the same. It is the subring consisting of fractions that can be written with denominator in W.
(f) We define a map from $W^{-1}R \to \overline{W}^{-1}\overline{R}$ by $\frac{r}{w} \mapsto \frac{\overline{r}}{\overline{w}}$. It is clear from the construction that this is a surjective homomorphism. Suppose that $\frac{r}{w}$ is in the kernel, so $\frac{\overline{r}}{\overline{w}} = \frac{\overline{0}}{\overline{1}}$. This means that there is some $\overline{v} \in \overline{W}$ such that $\overline{vr} = \overline{0}$; i.e., $vr \in {}^{W}0$ for some $v \in W$. Then there is some $u \in W$ such that $uvr = 0$, but $uv \in W$, so $\frac{r}{w} = \frac{0}{1}$ in $W^{-1}R$.

(2) Ideals in localizations: Let R be a ring and W a multiplicatively closed set.
(a) Use the Theorem to show that, if $f \in R$ is nonnilpotent, then

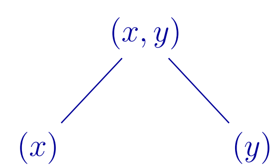
$$\mathrm{Spec}(R_f) \cong D(f) \subseteq \mathrm{Spec}(R).$$

(b) Use the Theorem to show that, if $\mathfrak{p} \subseteq R$ is prime, then

$$\mathrm{Spec}(R_\mathfrak{p}) \cong \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{q} \subseteq \mathfrak{p}\} =: \Lambda(\mathfrak{p}).$$

Deduce that $R_\mathfrak{p}$ is always a *local* ring.
(c) Draw[3] a picture of $\mathrm{Spec}(\frac{\mathbb{C}[X,Y]}{(XY)}_{(x,y)})$.
(d) Use Part (3) of the Lemma to show that every ideal of $W^{-1}R$ is of the form $W^{-1}I$ for some ideal $I \subseteq R$.
(e) Use Part (3) of the Lemma to show that any localization of a Noetherian ring is Noetherian.

(a) The condition $\mathfrak{p} \cap \{1, f, f^2, \dots\} = \varnothing$ is equivalent to $f \notin \mathfrak{p}$; i.e., $f \in D(\mathfrak{p})$.
(b) The condition $\mathfrak{q} \cap (R \smallsetminus \mathfrak{p}) = \varnothing$ is equivalent to $\mathfrak{q} \subseteq \mathfrak{p}$; i.e., $\mathfrak{q} \in \Lambda(\mathfrak{p})$. There is a unique maximal element in this set, namely $\mathfrak{p}$, so $R_\mathfrak{p}$ is local.
(c)



(d) Clear.
(e) Given an ideal of $W^{-1}R$, write it as $I(W^{-1}R)$ for some ideal I of R. Then $I = (f_1, \dots, f_t)$ by Noetherianity, whence $I(W^{-1}R)$ is generated by the images $\frac{f_1}{1}, \dots, \frac{f_t}{1}$.

(3) Examples of localizations
(a) Describe as concretely as possible the rings $\mathbb{Z}_2$ and $\mathbb{Z}_{(2)}$ as defined above.
(b) Describe as concretely as possible the rings $K[X]_X$ and $K[X]_{(X)}$.
(c) Describe as concretely as possible the rings $K[X,Y]_X$ and $K[X,Y]_{(X)}$.
(d) Describe as concretely as possible the rings $\left(\frac{K[X,Y]}{(XY)}\right)_x$ and $\left(\frac{K[X,Y]}{(XY)}\right)_{(x)}$.

[3] Recall that $\mathrm{Spec}(\frac{\mathbb{C}[X,Y]}{(XY)})$ consists of $\{(x), (y), (x, y-\alpha), (x-\beta, y) \mid \alpha, \beta \in \mathbb{C}\}$.

(e) Describe as concretely as possible $\left(\frac{K[X,Y]}{(X^2)}\right)_x$ and $\left(\frac{K[X,Y]}{(X^2)}\right)_{(x)}$.

> **(a)** $\mathbb{Z}_2 = \{a/b \in \mathbb{Q} \mid b = 2^n\}$ and $\mathbb{Z}_{(2)} = \{a/b \in \mathbb{Q} \mid 2 \nmid b\}$.
> **(b)** $K[X]_X = \{f/g \in K(X) \mid g = X^n\}$ and $K[X]_{(X)} = \{f/g \in K(X) \mid X \nmid g\}$.
> **(c)** $K[X,Y]_X = \{f/g \in K(X,Y) \mid g = X^n\}$
> and $K[X,Y]_{(X)} = \{f/g \in K(X,Y) \mid X \nmid g\}$.
> **(d)** $\left(\frac{K[X,Y]}{(XY)}\right)_x \cong K[X,X^{-1}]$ and $\left(\frac{K[X,Y]}{(XY)}\right)_{(x)} \cong K(Y)$.
> **(e)** $\left(\frac{K[X,Y]}{(X^2)}\right)_x \cong K[Y]$ and $\left(\frac{K[X,Y]}{(X^2)}\right)_{(x)} \cong K(Y)[X]/(X^2)$.

(4) Prove the Lemma and the Theorem.

> Lemma:
>
> (a) For the containment $\subseteq$, we have $\frac{a}{w} = \frac{a}{1}\frac{1}{w}$. For the other, given $\sum_i \frac{a_i}{1}\frac{r_i}{w_i}$, take $w = w_1 \cdots w_t$ and w_i' to be the product of all w's except w_i; then
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> $$\sum_i \frac{a_i}{1}\frac{r_i}{w_i} = \sum_i \frac{a_i}{1}\frac{w_i' r_i}{w} = \sum_i \frac{a_i w_i' r_i}{w} \in W^{-1}I.$$
>
> (b) We have $r \in W^{-1}I \cap R$ if and only if $\frac{r}{1} \in W^{-1}I$, so $\frac{r}{1} = \frac{a}{w}$ some $a \in I, w \in W$. Then there is some $u \in W$ such that $u(wr - a) = 0$, so $(uw)r \in I$, as claimed.
>
> (c) Let $j = \frac{r}{w} \in J$. Then $\frac{r}{1} = wj \in J \cap R$, $\frac{r}{w} = \frac{1}{w}\frac{r}{1} \in W^{-1}(J \cap R)$. Conversely, if $\frac{a}{w} \in W^{-1}(J \cap R)$ so $a \in J \cap R$, then $\frac{a}{1} \in J$, and $\frac{a}{w} = \frac{1}{w}\frac{a}{1} \in J$.
>
> (d) Let $\frac{a}{u}, \frac{b}{v} \in W^{-1}R$, and $\frac{ab}{uv} \in W^{-1}\mathfrak{p}$. Then there are some $w \in W$ and $p \in \mathfrak{p}$ such that $\frac{ab}{uv} = \frac{p}{w}$, so there is $t \in W$ with $t(wab - uvp) = 0$, so $(tw)ab \in \mathfrak{p}$. Since $W \cap \mathfrak{p} = \varnothing$, $tw \notin \mathfrak{p}$ so $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, and hence $\frac{a}{u} \in W^{-1}\mathfrak{p}$ or $\frac{b}{v} \in W^{-1}\mathfrak{p}$.
>
> Theorem: Suppose that $\mathfrak{q}$ is a prime ideal in $W^{-1}R$ and $\mathfrak{q} \cap R = \mathfrak{p}$. Then $W^{-1}\mathfrak{p} = W^{-1}(\mathfrak{q} \cap R) = \mathfrak{q}$. This shows that the only ideal (in particular, the only prime ideal) that contracts to $\mathfrak{p}$ is $W^{-1}\mathfrak{p}$, so this map is injective. Since $W^{-1}\mathfrak{p}$ is prime for any $\mathfrak{p} \cap W = \varnothing$, and is the bogus ideal otherwise, the image is exactly the primes with $\mathfrak{p} \cap W = \varnothing$. To see that it induces a homeomorphism onto its image, it suffices to show that the image of a closed set is closed. One checks from the definition that the image of $V(W^{-1}I)$ is $V(I) \cap \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \cap W = \varnothing\}$.

(5) Prove the following LEMMA: If V, W are multiplicatively closed sets, then $(VW)^{-1}R \cong (\frac{V}{1})^{-1}(W^{-1}R)$, where $(\frac{V}{1})^{-1}$ is the image of V in $W^{-1}R$.

> Check that the map $(r/w)/(v/1) \mapsto r/(wv)$ is an isomorphism: it is clearly a ring homomorphism, and clearly surjective. If $r/(wv)$ is zero, then there is some $u \in VW$ with $ur = 0$. We can write $u = st$ with $s \in V$ and $t \in W$, so $str = 0$. But this implies that $s(r/w) = 0$ in $W^{-1}R$ (because there is some $t \in W$ such that $str = 0$), and this means that $(r/w)/(v/1) = 0$.

(6) Minimal primes.
  (a) Let $\mathfrak{p}$ be a minimal prime of R. Show that for any $a \in \mathfrak{p}$, there is some $u \notin \mathfrak{p}$ and $n \geq 1$ such that $ua^n = 0$.
  (b) Show that the set of minimal[4] primes $\mathrm{Min}(R)$ with the induced topology from $\mathrm{Spec}(R)$ is Hausdorff.
  (c) Let $R = K[X_1, X_2, X_3, \dots]/(\{X_i X_j \mid i \neq j\})$. Describe $\mathrm{Min}(R)$ as a topological space.

[4] $\mathrm{Min}(R)$ denotes the set of primes of R that are minimal. This is the same as $\mathrm{Min}(0)$ in our notation of minimal primes of an ideal; this conflict of notation is standard.