DEFINITION: Let R be a ring and W a multiplicatively closed subset with $0 \notin W$. The **localization** $W^{-1}R$ is the ring with

- elements equivalence classes of $(r, w) \in R \times W$, with the class of (r, w) denoted as $\dfrac{r}{w}$.
- with equivalence relation $\dfrac{s}{u} = \dfrac{t}{v}$ if there is some $w \in W$ such that $w(sv - tu) = 0$,
- addition given by $\dfrac{s}{u} + \dfrac{t}{v} = \dfrac{sv + tu}{uv}$, and
- multiplication given by $\dfrac{s}{u}\dfrac{t}{v} = \dfrac{st}{uv}$.

(If $0 \in W$, then $W^{-1}R := 0$, which by our convention is not a ring.)

DEFINITION: Let R be a ring.

- If $f \in R$ is nonnilpotent[1], then $R_f := \{1, f, f^2, \dots\}^{-1}R$.
- If $\mathfrak{p} \subseteq R$ is a prime ideal then $R_{\mathfrak{p}} := (R \smallsetminus \mathfrak{p})^{-1}R$.
- The **total quotient ring** of R is $\mathrm{Frac}(R) := \{w \in R \mid w \text{ is a nonzerodivisor}\}^{-1}R$.

For a ring R, multiplicative set $W \not\ni 0$, and an ideal I, we define

$$W^{-1}I := \left\{ \frac{a}{w} \in W^{-1}R \mid a \in I \right\}.$$

THEOREM: Let R be a ring and W be a multiplicatively closed subset. Then the map induced on Spec corresponding to the natural map $R \to W^{-1}R$ yields a homeomorphism into its image:

$$\mathrm{Spec}(W^{-1}R) \cong \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \cap W = \varnothing\}.$$

LEMMA: Let R be a ring and W be a multiplicatively closed subset.

(1) For any ideal $I \subseteq R$, $W^{-1}I = I(W^{-1}R)$.
(2) For any ideal $I \subseteq R$, $W^{-1}I \cap R = \{r \in R \mid \exists w \in W : wr \in I\}$.
(3) For any ideal $J \subseteq W^{-1}R$, $W^{-1}(J \cap R) = J$.
(4) For any prime ideal $\mathfrak{p} \subseteq R$ with[2] $\mathfrak{p} \cap W = \varnothing$, $W^{-1}\mathfrak{p}$ is prime.

(1) Computing localizations
- **(a)** What is the natural ring homomorphism $R \to W^{-1}R$?
- **(b)** Show that the kernel of $R \to W^{-1}R$ is ${}^{W}0 := \{r \in R \mid \exists w \in W : wr = 0\}$.
- **(c)** If every element of W is a nonzerodivisor, explain why the equivalence relation on $W^{-1}R$ simplifies to $\frac{s}{u} = \frac{t}{v}$ if and only if $sv = tu$.
- **(d)** If R is a domain, explain why $\mathrm{Frac}(R)$ is the usual fraction field of R.
- **(e)** If R is a domain, explain why $W^{-1}R$ is a subring of the fraction field of R. Which subring?
- **(f)** Let $\overline{R} = R/{}^{W}0$ and $\overline{W}$ be the image of W in $\overline{R}$. Show that $W^{-1}R \cong \overline{W}^{-1}\overline{R}$.

[1] If f is nilpotent, $0 \in \{1, f, f^2, \dots\}$ so $R_f = 0$.
[2] If $W \cap \mathfrak{p} \ni a$, then $W^{-1}\mathfrak{p} \ni \frac{a}{a} = \frac{1}{1}$, so $W^{-1}\mathfrak{p} = W^{-1}R$ is the improper ideal!

(2) Ideals in localizations: Let R be a ring and W a multiplicatively closed set.

(a) Use the Theorem to show that, if $f \in R$ is nonnilpotent, then

$$\mathrm{Spec}(R_f) \cong D(f) \subseteq \mathrm{Spec}(R).$$

(b) Use the Theorem to show that, if $\mathfrak{p} \subseteq R$ is prime, then

$$\mathrm{Spec}(R_{\mathfrak{p}}) \cong \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{q} \subseteq \mathfrak{p}\} =: \Lambda(\mathfrak{p}).$$

Deduce that $R_{\mathfrak{p}}$ is always a *local* ring.

(c) Draw[3] a picture of $\mathrm{Spec}(\frac{\mathbb{C}[X,Y]}{(XY)}_{(x,y)})$.

(d) Use Part (3) of the Lemma to show that every ideal of $W^{-1}R$ is of the form $W^{-1}I$ for some ideal $I \subseteq R$.

(e) Use Part (3) of the Lemma to show that any localization of a Noetherian ring is Noetherian.

(3) Examples of localizations

(a) Describe as concretely as possible the rings $\mathbb{Z}_2$ and $\mathbb{Z}_{(2)}$ as defined above.

(b) Describe as concretely as possible the rings $K[X]_X$ and $K[X]_{(X)}$.

(c) Describe as concretely as possible the rings $K[X,Y]_X$ and $K[X,Y]_{(X)}$.

(d) Describe as concretely as possible the rings $\left(\frac{K[X,Y]}{(XY)}\right)_x$ and $\left(\frac{K[X,Y]}{(XY)}\right)_{(x)}$.

(e) Describe as concretely as possible $\left(\frac{K[X,Y]}{(X^2)}\right)_x$ and $\left(\frac{K[X,Y]}{(X^2)}\right)_{(x)}$.

(4) Prove the Lemma and the Theorem.

(5) Prove the following LEMMA: If V, W are multiplicatively closed sets, then $(VW)^{-1}R \cong (\frac{V}{1})^{-1}(W^{-1}R)$, where $(\frac{V}{1})^{-1}$ is the image of V in $W^{-1}R$.

(6) Minimal primes.

(a) Let $\mathfrak{p}$ be a minimal prime of R. Show that for any $a \in \mathfrak{p}$, there is some $u \notin \mathfrak{p}$ and $n \geq 1$ such that $ua^n = 0$.

(b) Show that the set of minimal[4] primes $\mathrm{Min}(R)$ with the induced topology from $\mathrm{Spec}(R)$ is Hausdorff.

(c) Let $R = K[X_1, X_2, X_3, \ldots]/(\{X_iX_j \mid i \neq j\})$. Describe $\mathrm{Min}(R)$ as a topological space.

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[3] Recall that $\mathrm{Spec}(\frac{\mathbb{C}[X,Y]}{(XY)})$ consists of $\{(x), (y), (x, y-\alpha), (x-\beta, y) \mid \alpha, \beta \in \mathbb{C}\}$.

[4] $\mathrm{Min}(R)$ denotes the set of primes of R that are minimal. This is the same as $\mathrm{Min}(0)$ in our notation of minimal primes of an ideal; this conflict of notation is standard.