DEFINITION: A ring is **local** if it has a unique maximal ideal. We write $(R, \mathfrak{m})$ for a local ring to denote the ring R and the maximal ideal $\mathfrak{m}$; we many also write $(R, \mathfrak{m}, k)$ to indicate the residue field $k := R/\mathfrak{m}$.

GENERAL NAK: Let R be a ring, I an ideal, and M be a finitely generated module. If $IM = M$, then there is some $a \in R$ such that $a \equiv 1 \mod I$ and $aM = 0$.

LOCAL NAK 1: Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated module. If $M = \mathfrak{m}M$, then $M = 0$.

LOCAL NAK 2: Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated module. Let N be a submodule of M. Then $M = N + \mathfrak{m}M$ if and only if $M = N$.

LOCAL NAK 3: Let $(R, \mathfrak{m}, k)$ be a local ring and M be a finitely generated module. Then a set of elements $S \subseteq M$ generates M if and only if the image of S in $M/\mathfrak{m}M$ generates $M/\mathfrak{m}M$ as a k-vector space.

DEFINITION: Let $(R, \mathfrak{m}, k)$ be a local ring and M be a finitely generated module. A set of elements S of M is a **minimal generating set** for M if the image of S in $M/\mathfrak{m}M$ is a basis for $M/\mathfrak{m}M$ as a k-vector space.

(1) Local rings.

(a) Show that for a ring R the following are equivalent:
- R is a local ring.
- The set of all nonunits forms an ideal.
- The set of all nonunits is closed under addition.

(b) Show that if A is a domain then $A[X]$ is *not* a local ring.

(c) Show that if K is a field, the power series ring $R = K[\![X_1, \dots, X_n]\!]$ is a local ring.

(d) Let $p \in \mathbb{Z}$ be a prime number, and $\mathbb{Z}_{(p)} \subseteq \mathbb{Q}$ be the set of rational numbers that can be written with denominator *not* a multiple of p. Show that $(\mathbb{Z}_{(p)}, p\mathbb{Z}_{(p)})$ is a local ring.

(e) Show that any quotient of a local ring is also a local ring.

(a) Since any element times a nonunit is a nonunit, the last two are equivalent. Recall that an element is a unit if and only if it is not in any maximal ideal. So, if $(R, \mathfrak{m})$ is local, the nonunits are the elements of $\mathfrak{m}$, which is an ideal; conversely, if the nonunits form an ideal, then this ideal must be the unique maximal ideal.

(b) X and $X + 1$ are nonunits, but $1 = (X + 1) - X$ is a unit.

(c) The set of nonunits is the elements with zero constant term, which is the ideal $(X_1, \dots, X_n)$.

(d) First, check that this is a ring. Then note that the units in this ring are the fractions a/b with $p \nmid a, b$, which is complement of the ideal $p\mathbb{Z}_{(p)}$.

(e) This follows from the Lattice Isomorphism Theorem.

(2) General NAK implies Local NAKs

(a) Show that General NAK implies Local NAK 1.

(b) Briefly[1] explain why Local NAK 1 implies Local NAK 2.
(c) Briefly[2] explain why Local NAK 2 implies Local NAK 3.
(d) Use Local NAK 3 to briefly explain why a minimal generating set is a generating set, and that, in this setting, any generating set contains a minimal generating set.

> **(a)** If $\mathfrak{m}M = M$, then by General NAK, there is some $a \in \mathfrak{m}$ such that $a \equiv 1 \bmod \mathfrak{m}$ and $aM = 0$. But a must be a unit, so $M = 0$!
> **(b)** Same as the graded case: apply NAK 1 to M/N.
> **(c)** Same as the graded case: apply NAK 2 to $N = \sum_{s\in S} Rs$.
> **(d)** Same as the graded case: a k-basis for $M/\mathfrak{m}M$ is a k-spanning set for $M/\mathfrak{m}M$, and any k-spanning set for $M/\mathfrak{m}M$ contains a k-basis.

(3) Proof of General NAK: Let $M = \sum_{i=1}^{n} Rm_i$. Set v to be the row vector $[m_1, \dots, m_n]$.
(a) Suppose that $IM = M$. Explain why there is an $n \times n$ matrix A with entries in I such that $vA = v$.
(b) Apply a TRICK and complete the proof.

> **(a)** Each m_i is an element of IM, so we can write $m_i = \sum_j b_j n_j$ with $n_j \in M$ and $b_j \in I$. We can then write n_j as a linear combination of the m_i's. Combining all together, we can write $m_i = \sum_j a_j m_j$ with $a_j \in I$. These linear combinations are the columns of a matrix A as desired.
> **(b)** By the Eigenvector trick, $\det(A - \mathbb{1})$ kills v, so kills M. Going mod I we have $\det(A - \mathbb{1}) \equiv \det(-\mathbb{1}) \equiv \pm 1$; up to sign, $a = \det(A - \mathbb{1})$ is the element we seek.

(4) Let $(R, \mathfrak{m})$ be a local ring, $f \in R$ not a unit, and M be a nonzero finitely generated module. Show that there is some element of M that is *not* a multiple of f.

> Suppose otherwise. Then $M = fM$. We have $f \in \mathfrak{m}$, so $M = fM \subseteq \mathfrak{m}M \subseteq M$, so $M = \mathfrak{m}M$. But by NAK, we then have $M = 0$, a contradiction.

(5) Applications of NAK.
(a) Let R be a ring and I be a finitely generated ideal. Show that if $I^2 = I$ then there is some idempotent e such that $I = (e)$.
(b) Find a counterexample to (a) if I is *not* assumed to be finitely generated.
(c) Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated module. Show that $\bigcap_{n\geq 1} \mathfrak{m}^n M = 0$.
(d) Find a counterexample to (c) if $(R, \mathfrak{m})$ is still Noetherian local but M is not finitely generated.
(e) Find a counterexample to (c) if $(R, \mathfrak{m})$ with $M = R$, $\mathfrak{m}$ is a maximal ideal, but R is not necessarily Noetherian and local.
(f) Let R be a Noetherian ring, and M a finitely generated module. Let $\phi : M \to M$ be a surjective R-module homomorphism. Show[3] that ϕ must also be injective.
(g) Let $(R, \mathfrak{m})$ be a local ring. Suppose that $R_{\text{red}} := R/\sqrt{0}$ is a domain, and that there is some $f \in R$ such that R/fR is reduced (and nonzero). Show that R is reduced (and hence a domain).

[1] Reuse an old argument in a similar setting.
[2] It's déjà vu all over again.
[3] Hint: Take a page from the 818 playbook and give M an $R[X]$-module structure.