§5.20: LOCAL RINGS AND NAK

DEFINITION: A ring is **local** if it has a unique maximal ideal. We write $(R, \mathfrak{m})$ for a local ring to denote the ring R and the maximal ideal $\mathfrak{m}$; we many also write $(R, \mathfrak{m}, k)$ to indicate the residue field $k := R/\mathfrak{m}$.

GENERAL NAK: Let R be a ring, I an ideal, and M be a finitely generated module. If $IM = M$, then there is some $a \in R$ such that $a \equiv 1 \bmod I$ and $aM = 0$.

LOCAL NAK 1: Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated module. If $M = \mathfrak{m}M$, then $M = 0$.

LOCAL NAK 2: Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated module. Let N be a submodule of M. Then $M = N + \mathfrak{m}M$ if and only if $M = N$.

LOCAL NAK 3: Let $(R, \mathfrak{m}, k)$ be a local ring and M be a finitely generated module. Then a set of elements $S \subseteq M$ generates M if and only if the image of S in $M/\mathfrak{m}M$ generates $M/\mathfrak{m}M$ as a k-vector space.

DEFINITION: Let $(R, \mathfrak{m}, k)$ be a local ring and M be a finitely generated module. A set of elements S of M is a **minimal generating set** for M if the image of S in $M/\mathfrak{m}M$ is a basis for $M/\mathfrak{m}M$ as a k-vector space.

(1) Local rings.
  (a) Show that for a ring R the following are equivalent:
    - R is a local ring.
    - The set of all nonunits forms an ideal.
    - The set of all nonunits is closed under addition.

  (b) Show that if A is a domain then $A[X]$ is *not* a local ring.
  (c) Show that if K is a field, the power series ring $R = K[\![X_1, \dots, X_n]\!]$ is a local ring.
  (d) Let $p \in \mathbb{Z}$ be a prime number, and $\mathbb{Z}_{(p)} \subseteq \mathbb{Q}$ be the set of rational numbers that can be written with denominator *not* a multiple of p. Show that $(\mathbb{Z}_{(p)}, p\mathbb{Z}_{(p)})$ is a local ring.
  (e) Show that any quotient of a local ring is also a local ring.

(2) General NAK implies Local NAKs
  (a) Show that General NAK implies Local NAK 1.
  (b) Briefly[1] explain why Local NAK 1 implies Local NAK 2.
  (c) Briefly[2] explain why Local NAK 2 implies Local NAK 3.
  (d) Use Local NAK 3 to briefly explain why a minimal generating set is a generating set, and that, in this setting, any generating set contains a minimal generating set.

(3) Proof of General NAK: Let $M = \sum_{i=1}^{n} Rm_i$. Set v to be the row vector $[m_1, \dots, m_n]$.
  (a) Suppose that $IM = M$. Explain why there is an $n \times n$ matrix A with entries in I such that $vA = v$.
  (b) Apply a TRICK and complete the proof.

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[1]Reuse an old argument in a similar setting.
[2]It's déjà vu all over again.

(4) Let $(R, \mathfrak{m})$ be a local ring, $f \in R$ not a unit, and M be a nonzero finitely generated module. Show that there is some element of M that is *not* a multiple of f.

(5) Applications of NAK.
   (a) Let R be a ring and I be a finitely generated ideal. Show that if $I^2 = I$ then there is some idempotent e such that $I = (e)$.
   (b) Find a counterexample to (a) if I is *not* assumed to be finitely generated.
   (c) Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated module. Show that $\bigcap_{n \geq 1} \mathfrak{m}^n M = 0$.
   (d) Find a counterexample to (c) if $(R, \mathfrak{m})$ is still Noetherian local but M is not finitely generated.
   (e) Find a counterexample to (c) if $(R, \mathfrak{m})$ with $M = R$, $\mathfrak{m}$ is a maximal ideal, but R is not necessarily Noetherian and local.
   (f) Let R be a Noetherian ring, and M a finitely generated module. Let $\phi : M \to M$ be a surjective R-module homomorphism. Show[3] that ϕ must also be injective.
   (g) Let $(R, \mathfrak{m})$ be a local ring. Suppose that $R_{\text{red}} := R/\sqrt{0}$ is a domain, and that there is some $f \in R$ such that R/fR is reduced (and nonzero). Show that R is reduced (and hence a domain).

[3]Hint: Take a page from the 818 playbook and give M an $R[X]$-module structure.