FORMAL NULLSTELLENSATZ: Let R be a ring, I an ideal, and $f \in R$. Then $V(f) \supseteq V(I)$ if and only if $f \in \sqrt{I}$.

COROLLARY 1: Let R be a ring. There is a bijection

$$\{\text{radical ideals in } R\} \longleftrightarrow \{\text{closed subsets of } \operatorname{Spec}(R)\}.$$

DEFINITION: Let R be a ring and I an ideal. A **minimal prime** of I is a prime $\mathfrak{p}$ that contains I, and is minimal among primes containing I. We write $\operatorname{Min}(I)$ for the set of minimal primes of I.

LEMMA: Every prime that contains I contains a minimal prime of I.

COROLLARY 2: Let R be a ring and I be an ideal. Then

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in \operatorname{Min}(I)} \mathfrak{p}.$$

DEFINITION: A subset W of a ring R is **multiplicatively closed** if $1 \in W$ and $u, v \in W$ implies $uv \in W$.

PROPOSITION: Let R be a ring and W be a multiplicatively closed subset. Then every ideal I such that $I \cap W = \varnothing$ is contained in a prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \cap W = \varnothing$.

(1) Proof of Formal Nullstellensatz and Corollaries.
- **(a)** Show the direction ($\Leftarrow$) of Formal Nullstellensatz.
- **(b)** Verify that $W = \{f^n \mid n \geq 0\}$ is a multiplicatively closed set. Then apply the Proposition to prove the direction ($\Rightarrow$) of Formal Nullstellesatz.
- **(c)** Prove Corollary 1.
- **(d)** Prove the Lemma.
- **(e)** Prove Corollary 2.
- **(f)** What does Corollary 2 say in the special case $I = (0)$?

- **(a)** Suppose that $f \in \sqrt{I}$, so $f^n \in I$. If $\mathfrak{p} \in V(I)$, then $I \subseteq \mathfrak{p}$, and $f^n \in \mathfrak{p}$ implies $f \in \mathfrak{p}$, so $\mathfrak{p} \in V(f)$.
- **(b)** Yes, it is a multiplicatively closed set. If $f \notin \sqrt{I}$, then $W \cap I = \varnothing$, so there is some prime $\mathfrak{p}$ such that $W \cap \mathfrak{p} = \varnothing$. In particular, $f \notin \mathfrak{p}$, so $V(f) \not\supseteq V(I)$.
- **(c)** We map a radical ideal I to the closed set $V(I)$. This is surjective since $V(J) = V(\sqrt{J})$. If I, J are distinct radical ideals, then take some $f \in J \setminus I$. Then $V(f)$ contains $V(I)$ but not $V(J)$, so $V(I) \neq V(J)$.
- **(d)** Usual Zorn's Lemma argument.
- **(e)** If $f \in \sqrt{I}$, then $f \in V(\mathfrak{p})$ for all $\mathfrak{p}$ containing I, so f is in every minimal prime of I. On the other hand, if f is in every minimal prime of I, then it is in every prime containing I, so $V(f) \supseteq V(I)$, which implies $f \in \sqrt{I}$.
- **(f)** An element is nilpotent if and only if it is in every minimal prime of the ring.

(2) Use the Formal Nullstellensatz to fill in the blanks:

$$f \text{ is nilpotent} \iff V(f) = \_\_\_\_ \iff D(f) = \_\_\_\_.$$

What property replaces "nilpotent" if you swap the blanks for V and D above?

$$f \text{ is nilpotent} \iff V(f) = \operatorname{Spec}(R) \iff D(f) = \varnothing.$$

The opposite property is unit.

(3) Prove[1] the Proposition.

Given an increasing union of ideals that don't intersect I, the union is an ideal and does not intersect I, so by Zorn's Lemma, there is an ideal maximal among those that don't intersect I; call it J. Let $ab \in J$ with $a, b \notin J$. Then $(J + (a)) \cap W$ and $(J + (b)) \cap W$ are nonempty. Say u, v are elements in the respective intersections. Then $u = j_1 + ar_1$ and $v = j_2 + br_2$, and $uv = j_1 j_2 + j_1 br_2 + j_2 ar_2 + abr_1 r_2 \in J$.

(4) Let R be a ring. Show[2] that $\operatorname{Spec}(R)$ is connected as a topological space if and only if $R \ncong S \times T$ for rings[3] S, T.

First, suppose that $R \cong S \times T$. Then any prime ideal of R is of the form $\mathfrak{p} \times T$ for $\mathfrak{p} \in \operatorname{Spec}(S)$ or $S \times \mathfrak{q}$ for $\mathfrak{q} \in \operatorname{Spec}(T)$. So, as sets, there is a bijection $\operatorname{Spec}(R) \leftrightarrow \operatorname{Spec}(S) \coprod \operatorname{Spec}(T)$. Moveover, this is a homeomorphism: the ideals in $S \times T$ are of the form $I \times J$, and $V(I \times J) \subseteq \operatorname{Spec}(S \times T)$ corresponds to $V(I) \coprod V(J) \subseteq \operatorname{Spec}(S) \coprod \operatorname{Spec}(T)$, so this is the disjoint union topology. In particular, $\operatorname{Spec}(S)$ and $\operatorname{Spec}(T)$ are form a disconnection.

From above, we know that $\operatorname{Spec}(S \times T) \cong \operatorname{Spec}(S) \coprod \operatorname{Spec}(T)$ so it suffices to show that $\operatorname{Spec}(R)$ disconnected implies that R has a nontrivial idempotent. Applying the definition of disconnected, there exists some closed sets $V(I), V(J)$ such that $V(I) \cup V(J) = \operatorname{Spec}(R)$ and $V(I) \cap V(J) = \varnothing$. Thus $\sqrt{I + J} = R$, so $I + J = R$ and $\sqrt{I \cap J} = \sqrt{0}$, so $I \cap J$ consists of nilpotents. By CRT, we have $R/(I \cap J) \cong R/I \times R/J$. Set $N = I \cap J$. We have that there is a nontrivial idempotent in R/N but $e, 1 - e \notin N$. So there is some $e \in R$ such that $e - e^2 \in N$ so $e^n(1-e)^n = 0$ for some n. Set $I' = (e^n)$ and $J' = (1 - e)^n$. We claim that $I' + J' = R$ and $I' \cap J' = 0$. Indeed, in R/I', $\overline{e}$ is nilpotent, so $\overline{1 - e}$ is a unit, as is $(1 - e)^n$. Thus, we can write $(1 - e)^n u = 1 + e^n f$ for some $u, f \in R$, and hence $1 \in I' + J'$; then $I' \cap J' = I'J' = 0$. By CRT we have $R \cong R/I' \times R/J'$. Finally, it remains to note that $I', J' \neq 0$ to see that this is proper: we have $0 \neq \overline{e} = \overline{e^2} = \cdots = \overline{e^n}$ in R/N, so we must have $e^n \neq 0$ and likewise $(1-e)^n \neq 0$.

[1] Hint: Take an ideal maximal among those that don't intersect W.

[2] Start with the $(\Rightarrow)$ direction. For the other direction, use CRT.

[3] Recall that the zero ring is not a ring.