FORMAL NULLSTELLENSATZ: Let R be a ring, I an ideal, and $f \in R$. Then $V(f) \supseteq V(I)$ if and only if $f \in \sqrt{I}$.

COROLLARY 1: Let R be a ring. There is a bijection

$$\{\text{radical ideals in } R\} \longleftrightarrow \{\text{closed subsets of } \operatorname{Spec}(R)\}.$$

DEFINITION: Let R be a ring and I an ideal. A **minimal prime** of I is a prime $\mathfrak{p}$ that contains I, and is minimal among primes containing I. We write $\operatorname{Min}(I)$ for the set of minimal primes of I.

LEMMA: Every prime that contains I contains a minimal prime of I.

COROLLARY 2: Let R be a ring and I be an ideal. Then

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in \operatorname{Min}(I)} \mathfrak{p}.$$

DEFINITION: A subset W of a ring R is **multiplicatively closed** if $1 \in W$ and $u, v \in W$ implies $uv \in W$.

PROPOSITION: Let R be a ring and W be a multiplicatively closed subset. Then every ideal I such that $I \cap W = \varnothing$ is contained in a prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \cap W = \varnothing$.

(1) Proof of Formal Nullstellensatz and Corollaries.
- **(a)** Show the direction ($\Leftarrow$) of Formal Nullstellensatz.
- **(b)** Verify that $W = \{f^n \mid n \geq 0\}$ is a multiplicatively closed set. Then apply the Proposition to prove the direction ($\Rightarrow$) of Formal Nullstellesatz.
- **(c)** Prove Corollary 1.
- **(d)** Prove the Lemma.
- **(e)** Prove Corollary 2.
- **(f)** What does Corollary 2 say in the special case $I = (0)$?

(2) Use the Formal Nullstellensatz to fill in the blanks:

$$f \text{ is nilpotent} \iff V(f) = \_\_\_\_ \iff D(f) = \_\_\_\_.$$

What property replaces "nilpotent" if you swap the blanks for V and D above?

(3) Prove[1] the Proposition.

(4) Let R be a ring. Show[2] that $\operatorname{Spec}(R)$ is connected as a topological space if and only if $R \not\cong S \times T$ for rings[3] S, T.

[1]Hint: Take an ideal maximal among those that don't intersect W.
[2]Start with the ($\Rightarrow$) direction. For the other direction, use CRT.
[3]Recall that the zero ring is not a ring.