§4.18: SPECTRUM OF A RING

DEFINITION: Let R be a ring, and $I \subseteq R$ an ideal of R.

- The **spectrum** of a ring R, denoted $\mathrm{Spec}(R)$, is the set of prime ideals of R.
- We set $V(I) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I \subseteq \mathfrak{p}\}$, the set of primes containing I.
- We set $D(I) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I \not\subseteq \mathfrak{p}\}$, the set of primes *not* containing I.
- More generally, for any subset $S \subseteq R$, we define $V(S)$ and $D(S)$ analogously.

DEFINITION/PROPOSITION: The collection $\{V(I) \mid I \text{ an ideal of } R\}$ is the collection of closed subsets of a topology on R, called the **Zariski topology**; equivalently, the open sets are $D(I)$ for I an ideal of R.

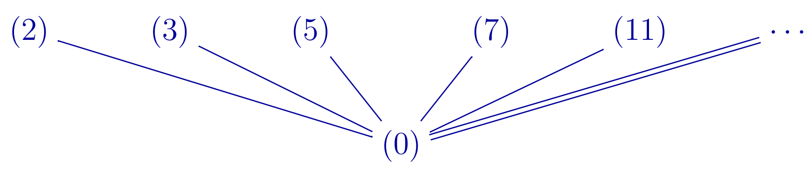
DEFINITION: Let $\phi : R \to S$ be a ring homomorphism. Then the **induced map on Spec** corresponding to ϕ is the map $\phi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ given by $\phi^*(\mathfrak{p}) := \phi^{-1}(\mathfrak{p})$.

LEMMA: Let $\mathfrak{p}$ be a prime ideal. Let I_λ, J be ideals.

(1) $\sum_\lambda I_\lambda \subseteq \mathfrak{p} \iff I_\lambda \subseteq \mathfrak{p}$ for all λ.
(2) $IJ \subseteq \mathfrak{p} \iff I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$
(3) $I \cap J \subseteq \mathfrak{p} \iff I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$
(4) $I \subseteq \mathfrak{p} \iff \sqrt{I} \subseteq \mathfrak{p}$

(1) The spectrum of some reasonably small rings.
(a) Let $R = \mathbb{Z}$ be the ring of integers.
(i) What are the elements of $\mathrm{Spec}(R)$? Be careful not to forget (0)!
(ii) Draw a picture $\mathrm{Spec}(R)$ (with $\cdots$ since you can't list everything) with a line going up from $\mathfrak{p}$ to $\mathfrak{q}$ if $\mathfrak{p} \subset \mathfrak{q}$.
(iii) Describe the sets $V(I)$ and $D(I)$ for any ideal I.
(b) Same questions for $R = K$ a field.
(c) Same questions for the polynomial ring $R = \mathbb{C}[X]$.
(d) Same questions[1] for the power series ring $R = K[\![X]\!]$ for a field K.

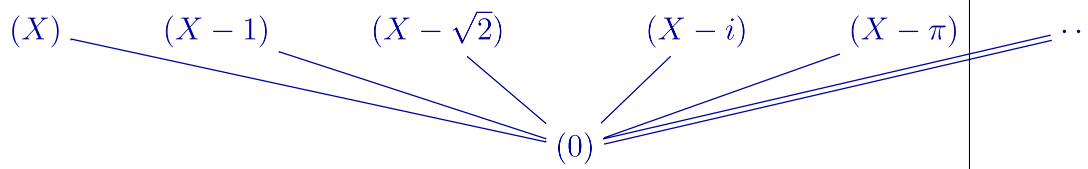
(a) The spectrum of $\mathbb{Z}$ is, as a poset:



The sets $D((n))$ are the whole space when $n = 1$, the empty set with $n = 0$, and any complement of finite union of things in the top row otherwise. The sets $V((n))$ are the whole space when $n = 0$, the empty set with $n = 1$, and any finite union of things in the top row otherwise.
(b) The spectrum of a field is just $\{(0)\}$.

[1]Spoiler: The only primes are (0) and (X). To prove it, show/recall that any nonzero series f can be written as $f = X^n u$ for some unit $u \in K[\![X]\!]$.

(c) The spectrum of $\mathbb{C}[X]$ is, as a poset:



For an element f, $V((f))$ corresponds to the irreducible factors of f. The sets $D((f))$ are the whole space when $f = 1$, the empty set with $f = 0$, and any complement of finite union of things in the top row otherwise. The sets $V((f))$ are the whole space when $f = 0$, the empty set with $f = 1$, and any finite union of things in the top row otherwise.

(d)

The sets V are $\varnothing$, $\{(X)\}$, and $\{(0),(X)\}$. The sets D are $\varnothing$, $\{(0)\}$, and $\{(0),(X)\}$.

(2) More Spectra.

(a) Let $R = \mathbb{C}[X, Y]$ be a polynomial ring in two variables. Find some maximal ideals, the zero ideal, and some primes that are neither. Draw a picture like the ones from the previous problem to illustrate some containments between these.

(b) Let R be a ring and I be an ideal. Use the Second Isomorphism Theorem to give a natural bijection between $\mathrm{Spec}(R/I)$ and $V(I)$.

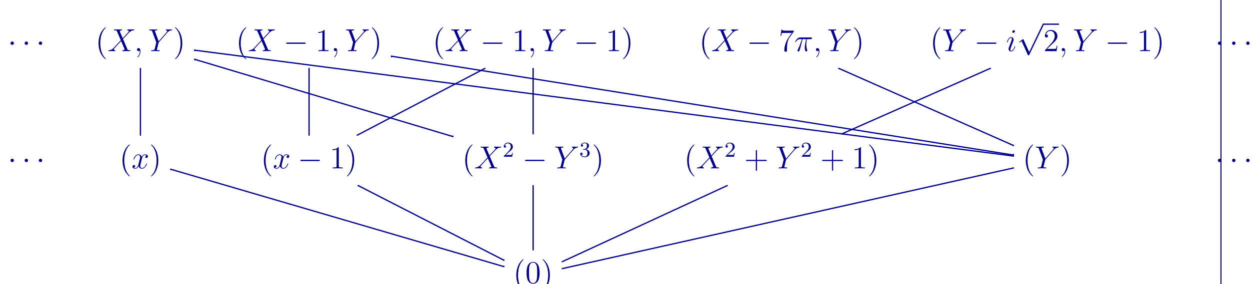
(c) Let $R = \dfrac{\mathbb{C}[X,Y]}{(XY)}$. Let $x = [X]$ and $y = [Y]$.

(i) Use the definition of prime ideal to show that $\mathrm{Spec}(R) = V(x) \cup V(y)$.

(ii) Use the previous problem to completely describe $V(x)$ and $V(y)$.

(iii) Give a complete description/picture of $\mathrm{Spec}(R)$.

(a)

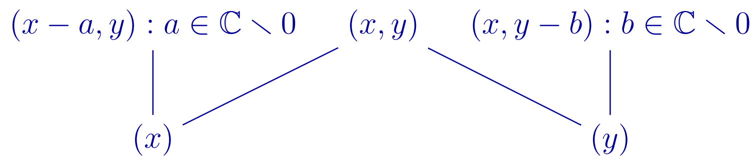


(b) $\mathfrak{p} \in V(I)$ maps to $\mathfrak{p}/I \in \mathrm{Spec}(R/I)$.

(c) **(i)** Since $xy = 0$, if $\mathfrak{p}$ is prime, we must have $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

(ii) $V(x) \cong \mathrm{Spec}(R/(x)) \cong \mathrm{Spec}(\mathbb{C}[Y])$ and $V(y) \cong \mathrm{Spec}(R/(y)) \cong \mathrm{Spec}(\mathbb{C}[X])$.

(iii)



(3) Let R be a ring.

(a) Show that for any subset S of R, $V(S) = V(I)$ where $I = (S)$.

(b) Translate the lemma to fill in the blanks:

$V(I)$ ___ $V(\sqrt{I})$	$D(I)$ ___ $D(\sqrt{I})$
$V(\sum_\lambda I_\lambda)$ _____ $V(I_\lambda)$	$D(\sum_\lambda I_\lambda)$ _____ $D(I_\lambda)$
$V(f_1, \ldots, f_n)$ __ $V(f_1)$ __ $\cdots$ __ $V(f_n)$	$D(f_1, \ldots, f_n)$ __ $D(f_1)$ __ $\cdots$ __ $D(f_n)$
$V(IJ)$___$V(I)$___$V(J)$	$D(IJ)$___$D(I)$___$D(J)$
$V(I \cap J)$___$V(I)$___$V(J)$	$D(I \cap J)$___$D(I)$___$D(J)$

(c) Use the above to verify that the Zariski topology indeed satisfies the axioms of a topology.

(a) This follows from definition of generating set of an ideal.

(b)

$V(I) = V(\sqrt{I})$	$D(I) = D(\sqrt{I})$
$V(\sum_\lambda I_\lambda) = \bigcap_\lambda V(I_\lambda)$	$D(\sum_\lambda I_\lambda) = \bigcup_\lambda D(I_\lambda)$
$V(f_1, \ldots, f_n) = V(f_1) \cap \cdots \cap V(f_n)$	$D(f_1, \ldots, f_n) = D(f_1) \cup \cdots \cup D(f_n)$
$V(IJ) = V(I) \cup V(J)$	$D(IJ) = D(I) \cap D(J)$
$V(I \cap J) = V(I) \cup V(J)$	$D(I \cap J) = D(I) \cap D(J)$

(c) The D's are closed under arbitrary unions and finite intersection; we also have $\mathrm{Spec}(R) = D(1)$ and $\varnothing = D(0)$.

(4) The induced map on Spec: Let $\phi : R \to S$ be a ring homomorphism.

(a) Show that for any prime ideal $\mathfrak{q} \subseteq S$, the ideal $\phi^*(\mathfrak{q}) = \phi^{-1}(\mathfrak{q})$ is a prime ideal of R.

(b) Show that for any ideal $I \in R$, we have

$$(\phi^*)^{-1}(V(I)) = V(IS) \text{ and } (\phi^*)^{-1}(D(I)) = D(IS).$$

(c) Show that ϕ^* is continuous.

(d) If $\phi : R \to R/I$ is quotient map, describe ϕ^*.

(a) $\phi^{-1}(\mathfrak{q})$ is the kernel of the map $R \xrightarrow{\phi} S \to S/\mathfrak{q}$, so by the First Isomorphism Theorem, $R/\phi^{-1}(\mathfrak{q})$ is isomorphic to a subring of $S/\mathfrak{q}$. Since $S/\mathfrak{q}$ is a domain, so is $R/\phi^{-1}(\mathfrak{q})$, so $\phi^{-1}(\mathfrak{q})$ is a prime ideal.
(b) Let $\mathfrak{q} \in \mathrm{Spec}(S)$. We claim that $\mathfrak{q} \in V(IS)$ if and only if $\mathfrak{p} := \phi^*(\mathfrak{q}) \in V(I)$, which shows both statements. Indeed, $\mathfrak{q} \in V(IS)$ is equivalent to $\mathfrak{q}$ contains IS. Since IS is generated by $\phi(I)$, this is equivalent to $\mathfrak{q} \supseteq \phi(I)$, which is equivalent to $\phi^{-1}(\mathfrak{q}) \supseteq I$. But this is the same as $\phi^{-1}(\mathfrak{q}) \in V(I)$.
(c) Follows from the previous.
(d) This corresponds to the embedding $V(I) \subseteq \mathrm{Spec}(R)$.

(5) Let R and S be rings. Describe $\mathrm{Spec}(R \times S)$ in terms of $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$.

(6) Properties of $\mathrm{Spec}(R)$.
 (a) Show that for any ring R, the space $\mathrm{Spec}(R)$ is compact.
 (b) Show that if $\mathrm{Spec}(R)$ is Hausdorff, then every prime of R is maximal.
 (c) Show that $\mathrm{Spec}(R) \cong \mathrm{Spec}(R/\sqrt{0})$.

(7) Let K be a field, and $R = \dfrac{K[X_1, X_2, \dots]}{(\{X_i - X_iX_j \mid 1 \leq i \leq j\})}$. Describe $\mathrm{Spec}(R)$ as a set and as a topological space.