

§4.18: SPECTRUM OF A RING

DEFINITION: Let R be a ring, and $I \subseteq R$ an ideal of R .

- The **spectrum** of a ring R , denoted $\text{Spec}(R)$, is the set of prime ideals of R .
- We set $V(I) := \{\mathfrak{p} \in \text{Spec}(R) \mid I \subseteq \mathfrak{p}\}$, the set of primes containing I .
- We set $D(I) := \{\mathfrak{p} \in \text{Spec}(R) \mid I \not\subseteq \mathfrak{p}\}$, the set of primes *not* containing I .
- More generally, for any subset $S \subseteq R$, we define $V(S)$ and $D(S)$ analogously.

DEFINITION/PROPOSITION: The collection $\{V(I) \mid I \text{ an ideal of } R\}$ is the collection of closed subsets of a topology on R , called the **Zariski topology**; equivalently, the open sets are $D(I)$ for I an ideal of R .

DEFINITION: Let $\phi : R \rightarrow S$ be a ring homomorphism. Then the **induced map on Spec** corresponding to ϕ is the map $\phi^* : \text{Spec}(S) \rightarrow \text{Spec}(R)$ given by $\phi^*(\mathfrak{p}) := \phi^{-1}(\mathfrak{p})$.

LEMMA: Let \mathfrak{p} be a prime ideal. Let I_λ, J be ideals.

- (1) $\sum_\lambda I_\lambda \subseteq \mathfrak{p} \iff I_\lambda \subseteq \mathfrak{p}$ for all λ .
- (2) $IJ \subseteq \mathfrak{p} \iff I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$
- (3) $I \cap J \subseteq \mathfrak{p} \iff I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$
- (4) $I \subseteq \mathfrak{p} \iff \sqrt{I} \subseteq \mathfrak{p}$

(1) The spectrum of some reasonably small rings.

(a) Let $R = \mathbb{Z}$ be the ring of integers.

(i) What are the elements of $\text{Spec}(R)$? Be careful not to forget (0)!

(ii) Draw a picture $\text{Spec}(R)$ (with \dots since you can't list everything) with a line going up from \mathfrak{p} to \mathfrak{q} if $\mathfrak{p} \subset \mathfrak{q}$.

(iii) Describe the sets $V(I)$ and $D(I)$ for any ideal I .

(b) Same questions for $R = K$ a field.

(c) Same questions for the polynomial ring $R = \mathbb{C}[X]$.

(d) Same questions¹ for the power series ring $R = K[[X]]$ for a field K .

(2) More Spectra.

(a) Let $R = \mathbb{C}[X, Y]$ be a polynomial ring in two variables. Find some maximal ideals, the zero ideal, and some primes that are neither. Draw a picture like the ones from the previous problem to illustrate some containments between these.

(b) Let R be a ring and I be an ideal. Use the Second Isomorphism Theorem to give a natural bijection between $\text{Spec}(R/I)$ and $V(I)$.

(c) Let $R = \frac{\mathbb{C}[X, Y]}{(XY)}$. Let $x = [X]$ and $y = [Y]$.

(i) Use the definition of prime ideal to show that $\text{Spec}(R) = V(x) \cup V(y)$.

(ii) Use the previous problem to completely describe $V(x)$ and $V(y)$.

(iii) Give a complete description/picture of $\text{Spec}(R)$.

¹Spoiler: The only primes are (0) and (X) . To prove it, show/recall that any nonzero series f can be written as $f = X^n u$ for some unit $u \in K[[X]]$.

(3) Let R be a ring.

(a) Show that for any subset S of R , $V(S) = V(I)$ where $I = (S)$.

(b) Translate the lemma to fill in the blanks:

$$V(I) \text{ ____ } V(\sqrt{I})$$

$$D(I) \text{ ____ } D(\sqrt{I})$$

$$V\left(\sum_{\lambda} I_{\lambda}\right) \text{ ________ } V(I_{\lambda})$$

$$D\left(\sum_{\lambda} I_{\lambda}\right) \text{ ________ } D(I_{\lambda})$$

$$V(f_1, \dots, f_n) \text{ ____ } V(f_1) \text{ ____ } \dots \text{ ____ } V(f_n)$$

$$D(f_1, \dots, f_n) \text{ ____ } D(f_1) \text{ ____ } \dots \text{ ____ } D(f_n)$$

$$V(IJ) \text{ ____ } V(I) \text{ ____ } V(J)$$

$$D(IJ) \text{ ____ } D(I) \text{ ____ } D(J)$$

$$V(I \cap J) \text{ ____ } V(I) \text{ ____ } V(J)$$

$$D(I \cap J) \text{ ____ } D(I) \text{ ____ } D(J)$$

(c) Use the above to verify that the Zariski topology indeed satisfies the axioms of a topology.

(4) The induced map on Spec: Let $\phi : R \rightarrow S$ be a ring homomorphism.

(a) Show that for any prime ideal $\mathfrak{q} \subseteq S$, the ideal $\phi^*(\mathfrak{q}) = \phi^{-1}(\mathfrak{q})$ is a prime ideal of R .

(b) Show that for any ideal $I \in R$, we have

$$(\phi^*)^{-1}(V(I)) = V(IS) \text{ and } (\phi^*)^{-1}(D(I)) = D(IS).$$

(c) Show that ϕ^* is continuous.

(d) If $\phi : R \rightarrow R/I$ is quotient map, describe ϕ^* .

(5) Let R and S be rings. Describe $\text{Spec}(R \times S)$ in terms of $\text{Spec}(R)$ and $\text{Spec}(S)$.

(6) Properties of $\text{Spec}(R)$.

(a) Show that for any ring R , the space $\text{Spec}(R)$ is compact.

(b) Show that if $\text{Spec}(R)$ is Hausdorff, then every prime of R is maximal.

(c) Show that $\text{Spec}(R) \cong \text{Spec}(R/\sqrt{0})$.

(7) Let K be a field, and $R = \frac{K[X_1, X_2, \dots]}{(\{X_i - X_i X_j \mid 1 \leq i \leq j\})}$. Describe $\text{Spec}(R)$ as a set and as a topological space.