STRONG NULLSTELLENSATZ: Let K be an algebraically closed field, and $R = K[X_1, \dots, X_n]$ be a polynomial ring. Let $I \subseteq R$ be an ideal and $f \in R$ a polynomial. Then

$$f \text{ vanishes at every point of } \mathcal{Z}(I) \text{ if and only if } f \in \sqrt{I}.$$

DEFINITION: Let K be a field and $R = K[X_1, \dots, X_n]$. A **subvariety** of K^n is a set of the form $\mathcal{Z}(S)$ for some set of polynomials $S \subseteq R$; i.e., a solution set of some system of polynomial equations.

COROLLARY: Let K be an algebraically closed field. There is a bijection

$$\{\text{radical ideals in } K[X_1, \dots, X_n]\} \longleftrightarrow \{\text{subvarieties of } K^n\}.$$

(1) Proof of Strong Nullstellensatz:
  - **(a)** Show that $\mathcal{Z}(I) = \mathcal{Z}(\sqrt{I})$, and deduce the ($\Leftarrow$) direction.
  - **(b)** Let Y be an extra indeterminate. Show that f vanishes on $\mathcal{Z}(I)$ implies that

$$\mathcal{Z}\big(I + (Yf - 1)\big) = \varnothing \quad \text{in } K^{n+1}.$$

  - **(c)** What does the Nullstellensatz have to say about that?
  - **(d)** Apply the R-algebra homomorphism $\phi : R[Y] \to \mathrm{frac}(R)$ given by $\phi(Y) = \frac{1}{f}$ and clear denominators.

(2) Strong Nullstellensatz warmup:
  - **(a)** Consider the ideal $I = (X^2 + Y^2) \in \mathbb{R}[X, Y]$ and $f = X$. Discuss the hypotheses and conclusion of Strong Nullstellensatz in this example.
  - **(b)** Show that[1] no power of $F = X^2 + Y^2 + Z^2$ is in the ideal

$$I = (X^3 - Y^2 Z, Y^7 - XZ^3, 3X^5 - XYZ - 2Z^{19}) \quad \text{in the ring} \quad \mathbb{C}[X, Y, Z].$$

(3) Prove the Corollary.

(4) Let $R = \mathbb{C}[T]$ be a polynomial ring. In this problem, we will show that the ideal of $\mathbb{C}$-algebraic relations on the elements $\{T^2, T^3, T^4\}$ is $I = (X_1^2 - X_3, X_2^2 - X_1 X_3)$.
  - **(a)** Let $\phi : \mathbb{C}[X_1, X_2, X_3] \to \mathbb{C}[T]$ be the $\mathbb{C}$-algebra map $X_1 \mapsto T^2, X_2 \mapsto T^3, X_3 \mapsto T^4$. Show that $I \subseteq \ker(\phi)$.
  - **(b)** Show that $\mathcal{Z}(I) \subseteq \{(\lambda^2, \lambda^3, \lambda^4) \in \mathbb{C}^3 \mid \lambda \in \mathbb{C})\} \subseteq \mathcal{Z}(\ker(\phi))$, and deduce that $\ker(\phi) \subseteq \sqrt{I}$.
  - **(c)** Show that I is prime[2], and complete the proof.

(5) Let K be an algebraically closed field and $R = K\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ be a polynomial ring. Use the Strong Nullstellensatz to show that any polynomial $F(X_{11}, X_{12}, X_{21}, X_{22})$ that vanishes on every matrix of rank at most one is a multiple of $\det \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$.

[1]Hint: You just need to find one point. *One*, *one*, *one*...

[2]Show $\mathbb{C}[X_1, X_2, X_3]/I$ is a domain by simplifying the quotient.

(6) We say that a subvariety of K^n is **irreducible** if it cannot be written as a union of two proper subvarities. Show that the bijection from the Corollary restricts to a bijection

$$\{\text{prime ideals in } K[X_1, \ldots, X_n]\} \longleftrightarrow \{\text{irreducible subvarieties of } K^n\}.$$

(7) Use the Strong Nullstellensatz to show that, in a finitely generated algebra over an algebrically closed field, every radical ideal can be written as an intersection of maximal ideals.