NOETHER NORMALIZATION: Let K be a field, and R be a finitely-generated K-algebra. Then there exists a finite[1] set of elements $f_1, \dots, f_m \in R$ that are algebraically independent over K such that $K[f_1, \dots, f_m] \subseteq R$ is module-finite; equivalently, there is a module-finite injective K-algebra map from a polynomial ring $K[X_1, \dots, X_m] \hookrightarrow R$. Such a ring S is called a **Noether normalization** for R.

LEMMA: Let A be a ring, and $F \in R := A[X_1, \dots, X_n]$ be a nonzero polynomial. Then there exists an A-algebra automorphism ϕ of R such that $\phi(F)$, viewed as a polynomial in X_n with coefficients in $A[X_1, \dots, X_{n-1}]$, has top degree term aX_n^t for some $a \in A \smallsetminus 0$ and $t \geq 0$.

- If $A = K$ is an infinite field, one can take $\phi(X_n) = X_n$ and $\phi(X_i) = X_i + \lambda_i X_n$ for some $\lambda_1, \dots, \lambda_{n-1} \in K$.
- In general, if the top degree of F (with respect to the standard grading) is D, one can take $\phi(X_n) = X_n$ and $\phi(X_i) = X_i + X_n^{D^{n-i}}$ for $i < n$.

ZARISKI'S LEMMA: An algebra-finite extension of fields is module-finite.

USEFUL VARIATIONS ON NOETHER NORMALIZATION:

- NN FOR DOMAINS: Let $A \subseteq R$ be an algebra-finite inclusion of domains[2]. Then there exists $a \in A \smallsetminus 0$ and $f_1, \dots, f_m \in R[1/a]$ that are algebraically independent over $A[1/a]$ such that $A[1/a][f_1, \dots, f_m] \subseteq R[1/a]$ is module-finite.
- GRADED NN: Let K be an infinite field, and R be a standard graded K-algebra. Then there exist algebraically independent elements $L_1, \dots, L_m \in R_1$ such that $K[L_1, \dots, L_m] \subseteq R$ is module-finite.
- NN FOR POWER SERIES: Let K be an infinite field, and $R = K[\![X_1, \dots, X_n]\!]/I$. Then there exists a module-finite injection $K[\![Y_1, \dots, Y_m]\!] \hookrightarrow R$ for some power series ring in m variables.

(1) Examples of Noether normalizations: Let K be a field.

(a) Show that $K[x, y]$ is a Noether normalization of $R = \dfrac{K[X, Y, Z]}{(X^3 + Y^3 + Z^3)}$, where x, y are the classes of X and Y in R, respectively.

(b) Show that $K[x]$ is *not* a Noether normalization of $R = \dfrac{K[X, Y]}{(XY)}$. Then show that $K[x + y] \subseteq R$ *is* a Noether normalization.

(c) Show that $K[X^4, Y^4]$ is a Noether normalization for $R = K[X^4, X^3Y, XY^3, Y^4]$.

(a) From the equation $z^3 + x^3 + y^3 = 0$, we have $K[x, y] \subseteq R$ is integral, and since z generates as an algebra, hence module-finite. We need to check that x, y are algebraically independent in R. Suppose that $p(x, y) = 0$ in R, so $p(X, Y) \in$

[1] Possibly empty!

[2] The assumption that R is a domain is actually not necessary, but can't quite state the general statement yet. We assume that R is a domain so that there is fraction field of R in which to take $R[1/a]$.

$(X^3 + Y^3 + Z^3)$ in $K[X,Y,Z]$. By considering $K[X,Y,Z] = K[X,Y][Z]$ as polynomials in Z, the Z-degree of such a p, which forces $p = 0$. Thus x, y are algebraically independent.

(b) y is not integral over $K[x]$: this would imply $Y^n + a_1(X)Y^{n-1} + \cdots a_n(X) = XYb(X,Y)$ in $K[X,Y]$, but no monomial from any term can cancel Y^n. Alternatively, if the inclusion is module-finite, go mod x to get $K \subseteq K[X,Y]/(XY,X) = K[Y]$ module-finite, which it isn't.

(c) It is easy to check that X^4, Y^4 are algebraically independent, and $(X^3Y)^4 = (X^4)^3Y^4$, $(XY^3)^4 = X^4(Y^4)^3$ give integral dependence relations for the algebra generators.

(2) Use Noether Normalization[3] to prove Zariski's Lemma.

Let $K \subseteq L$ be an algebra-finite extension of fields. Take a NN of L: say $K \subseteq K[\ell_1, \dots, \ell_t] \subseteq L$, with ℓ_i algebraically independent and $R := K[\ell_1, \dots, \ell_t] \subseteq L$ module-finite and a fortiori integral. From the Integral Extensions worksheet, since L and R are domains, the extension is integral, and L is a field, we know that R is a field. This means that $t = 0$, so $K \subseteq L$ is module-finite.

(3) Proof of Noether Normalization (using the Lemma): Proceed by induction on the number of generators of R as a K-algebra; write $R = K[r_1, \dots, r_n]$.

(a) Deal with the base case $n = 0$.

(b) For the inductive step, first do the case that $r_1, \dots, r_n$ are algebraically independent over K.

(c) Let $\alpha : K[X_1, \dots, X_n] \to R$ be the K-algebra homomorphism such that $\alpha(X_i) = r_i$, and let ϕ be a K-algebra automorphism of $K[X_1, \dots, X_n]$. Let $r'_i = \alpha(\phi(X_i))$ for each i. Explain[4] why $R = K[r'_1, \dots, r'_n]$, and for any K-algebra relation F on $r_1, \dots, r_n$, the polynomial $\phi^{-1}(F)$ is a K-algebra relation on $r'_1, \dots, r'_n$.

(d) Use the Lemma to find a K-subalgebra R' of R with $n-1$ generators such that the inclusion $R' \subseteq R$ is module-finite.

(e) Conclude the proof.

(a) This means that R is a quotient of K, but K is a field, so $R = K$; the identity map is module-finite.

(b) If we have an algebraically independent set of generators for R, then R works: the identity map is module-finite.

(c) First we claim that $R = K[r'_1, \dots, r'_n]$: indeed, the map $\alpha' = \alpha \circ \phi$ is the K-algebra map that sends X_i to r'_i, and since α and ϕ are surjective, α' is surjective, verifying the claim. The relations on the r'_i are of the elements of the kernel of α'; if F is a relation on the originals, then $\alpha(F) = 0$, so $\alpha'(\phi^{-1}(F)) = 0$ as well.

[3] and a suitable fact about integral extensions...

[4] Say α' is the K-algebra map given by $\alpha'(X_i) = r'_i$. Observe that $\alpha' = \alpha \circ \phi$. Why is this surjective?

(d) Take a map ϕ as in the Lemma, and n generators $r_1, \dots, r_n$. Set $r'_i = \phi^{-1}(r_i)$. By the previous part, these generate, and there is a relation on these that is monic in X_n, so $R' = K[r'_1, \dots, r'_{n-1}] \subseteq R$ is module-finite.

(e) Apply IH to R' to get $K[f_1, \dots, f_t] \subseteq R'$ with f_i alg indep't and the inclusion module-finite. Then $K[f_1, \dots, f_t]$ is a Noether normalization.

(4) Proof of the "general case" of the Lemma:
- (a) Where do "base D expansions" fit in this picture?
- (b) Consider the automorphism ϕ from the general case of the Lemma. Show that for a monomial, we have $\phi(aX_1^{d_1} \cdots X_n^{d_n})$ is a polynomial with unique highest degree term $aX_n^{d_1 D^{n-1} + d_2 D^{n-2} + \cdots + d_n}$.
- (c) Can two monomials μ, ν in F, have $\phi(\mu)$ and $\phi(\nu)$ with the same highest degree term?
- (d) Complete the proof.

(5) Variations on NN.
- (a) Adapt the proof of NN to show Graded NN.
- (b) Adapt the proof of NN to show NN for domains.
- *(c)* Adapt the proof of NN to show NN for power series.