§4.15: NOETHER NORMALIZATION AND ZARISKI'S LEMMA

NOETHER NORMALIZATION: Let K be a field, and R be a finitely-generated K-algebra. Then there exists a finite[1] set of elements $f_1, \dots, f_m \in R$ that are algebraically independent over K such that $K[f_1, \dots, f_m] \subseteq R$ is module-finite; equivalently, there is a module-finite injective K-algebra map from a polynomial ring $K[X_1, \dots, X_m] \hookrightarrow R$. Such a ring S is called a **Noether normalization** for R.

LEMMA: Let A be a ring, and $F \in R := A[X_1, \dots, X_n]$ be a nonzero polynomial. Then there exists an A-algebra automorphism ϕ of R such that $\phi(F)$, viewed as a polynomial in X_n with coefficients in $A[X_1, \dots, X_{n-1}]$, has top degree term aX_n^t for some $a \in A \smallsetminus 0$ and $t \geq 0$.

- If $A = K$ is an infinite field, one can take $\phi(X_n) = X_n$ and $\phi(X_i) = X_i + \lambda_i X_n$ for some $\lambda_1, \dots, \lambda_{n-1} \in K$.
- In general, if the top degree of F (with respect to the standard grading) is D, one can take $\phi(X_n) = X_n$ and $\phi(X_i) = X_i + X_n^{D^{n-i}}$ for $i < n$.

ZARISKI'S LEMMA: An algebra-finite extension of fields is module-finite.

USEFUL VARIATIONS ON NOETHER NORMALIZATION:

- NN FOR DOMAINS: Let $A \subseteq R$ be an algebra-finite inclusion of domains[2]. Then there exists $a \in A \smallsetminus 0$ and $f_1, \dots, f_m \in R[1/a]$ that are algebraically independent over $A[1/a]$ such that $A[1/a][f_1, \dots, f_m] \subseteq R[1/a]$ is module-finite.
- GRADED NN: Let K be an infinite field, and R be a standard graded K-algebra. Then there exist algebraically independent elements $L_1, \dots, L_m \in R_1$ such that $K[L_1, \dots, L_m] \subseteq R$ is module-finite.
- NN FOR POWER SERIES: Let K be an infinite field, and $R = K[\![X_1, \dots, X_n]\!]/I$. Then there exists a module-finite injection $K[\![Y_1, \dots, Y_m]\!] \hookrightarrow R$ for some power series ring in m variables.

(1) Examples of Noether normalizations: Let K be a field.

(a) Show that $K[x, y]$ is a Noether normalization of $R = \dfrac{K[X, Y, Z]}{(X^3 + Y^3 + Z^3)}$, where x, y are the classes of X and Y in R, respectively.

(b) Show that $K[x]$ is *not* a Noether normalization of $R = \dfrac{K[X, Y]}{(XY)}$. Then show that $K[x + y] \subseteq R$ *is* a Noether normalization.

(c) Show that $K[X^4, Y^4]$ is a Noether normalization for $R = K[X^4, X^3Y, XY^3, Y^4]$.

(2) Use Noether Normalization[3] to prove Zariski's Lemma.

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[1] Possibly empty!

[2] The assumption that R is a domain is actually not necessary, but can't quite state the general statement yet. We assume that R is a domain so that there is fraction field of R in which to take $R[1/a]$.

[3] and a suitable fact about integral extensions…

(3) Proof of Noether Normalization (using the Lemma): Proceed by induction on the number of generators of R as a K-algebra; write $R = K[r_1, \dots, r_n]$.

(a) Deal with the base case $n = 0$.

(b) For the inductive step, first do the case that $r_1, \dots, r_n$ are algebraically independent over K.

(c) Let $\alpha : K[X_1, \dots, X_n] \to R$ be the K-algebra homomorphism such that $\alpha(X_i) = r_i$, and let ϕ be a K-algebra automorphism of $K[X_1, \dots, X_n]$. Let $r'_i = \alpha(\phi(X_i))$ for each i. Explain[4] why $R = K[r'_1, \dots, r'_n]$, and for any K-algebra relation F on $r_1, \dots, r_n$, the polynomial $\phi^{-1}(F)$ is a K-algebra relation on $r'_1, \dots, r'_n$.

(d) Use the Lemma to find a K-subalgebra R' of R with $n - 1$ generators such that the inclusion $R' \subseteq R$ is module-finite.

(e) Conclude the proof.

(4) Proof of the "general case" of the Lemma:

(a) Where do "base D expansions" fit in this picture?

(b) Consider the automorphism ϕ from the general case of the Lemma. Show that for a monomial, we have $\phi(aX_1^{d_1} \cdots X_n^{d_n})$ is a polynomial with unique highest degree term $aX_n^{d_1 D^{n-1} + d_2 D^{n-2} + \cdots + d_n}$.

(c) Can two monomials μ, ν in F, have $\phi(\mu)$ and $\phi(\nu)$ with the same highest degree term?

(d) Complete the proof.

(5) Variations on NN.

(a) Adapt the proof of NN to show Graded NN.

(b) Adapt the proof of NN to show NN for domains.

(c) Adapt the proof of NN to show NN for power series.

[4] Say α' is the K-algebra map given by $\alpha'(X_i) = r'_i$. Observe that $\alpha' = \alpha \circ \phi$. Why is this surjective?