DEFINITION: Let R be a ring and I be an ideal. The **Rees ring** of I is the $\mathbb{N}$-graded R-algebra

$$R[IT] := \bigoplus_{d\geq 0} I^d T^d = R \oplus IT \oplus I^2T^2 \oplus \cdots$$

with multiplication determined by $(aT^d)(bT^e) = abT^{d+e}$ for $a \in I^d$, $b \in I^e$ (and extended by the distributive law for nonhomogeneous elements). Here I^n means the nth power of the ideal I in R, and t is an indeterminate. Equivalently, $R[IT]$ is the R-subalgebra of the polynomial ring $R[T]$ generated by IT, with $R[T]$ is given the standard grading $R[T]_d = R \cdot T^d$.

DEFINITION: Let R be a ring and I be an ideal. The **associated graded ring** of I is the $\mathbb{N}$-graded ring

$$\mathrm{gr}_I(R) := \bigoplus_{d\geq 0} (I^d/I^{d+1})T^d = R/I \oplus (I/I^2)T \oplus (I^2/I^3)T^2 \oplus \cdots$$

with multiplication determined by $(a + I^{d+1}T^d)(b + I^{e+1}T^e) = ab + I^{d+e+1}\,T^{d+e}$ for $a \in I^d$, $b \in I^e$ (and extended by the distributive law). For an element $r \in R$, its **initial form** in $\mathrm{gr}_I(R)$ is

$$r^* := \begin{cases} (r + I^{d+1})T^d & \text{if } r \in I^d \smallsetminus I^{d+1} \\ 0 & \text{if } r \in \bigcap_{n\geq 0} I^n. \end{cases}$$

ARTIN-REES LEMMA: Let R be a Noetherian ring, I an ideal of R, M a finitely generated module, and $N \subseteq M$ a submodule. Then there is a constant[1] $c \geq 0$ such that for all $n \geq c$, we have $I^n M \cap N \subseteq I^{n-c}N$.

(1) Warmup with Rees rings:
- **(a)** Let R be a ring and I be an ideal. Show that if $I = (a_1, \dots, a_n)$, then $R[It] = R[a_1t, \dots, a_nt]$.
- **(b)** Let K be a field, $R = K[X, Y]$ and $I = (X, Y)$. Find K-algebra generators for $R[It]$, and find a relation on these generators.

- **(a)** This follows from the Theorem we showed last time: given a (finite, though this isn't necessary) set of homogeneous elements that generate R_+ as an ideal, these elements generate R as an R_0-algebra.
- **(b)** The elements X, Y, XT, YT generate. A relation is $X(YT) - Y(XT)$, or $X_1X_4 - X_2X_3$ in dummy variables. In fact, this is a defining set of relations.

(2) Warmup with associated graded rings:
- **(a)** Convince yourself that the multiplication given in the definition of $\mathrm{gr}_I(R)$ is well-defined. After doing this, do *not* use coset notation for elements of $\mathrm{gr}_I(R)$ and instead write a typical homogeneous element as something like $\overline{r}\,T^d$.
- **(b)** Let K be a field, $R = K[X, Y]$, and $\mathfrak{m} = (X, Y)$. Show that $\mathrm{gr}_{\mathfrak{m}}(R)_d \cong R_d$ as K-vector spaces, and construct a ring isomorphism $\mathrm{gr}_{\mathfrak{m}}(R) \cong R$.
- **(c)** For the same R, show that the map $R \to \mathrm{gr}_{\mathfrak{m}}(R)$ given by $r \mapsto r^*$ is *not* a ring homomorphism.
- **(d)** Let K be a field, $R = K[\![X, Y]\!]$, and $\mathfrak{m} = (X, Y)$. Show[2] that $\mathrm{gr}_{\mathfrak{m}}(R) \cong K[X, Y]$.

[1] The constant c depends on I, M, and N but works for all t.

[2] Yes, the brackets changed. This is not a typo!

(e) What happens in (b) and (d) if we have n variables instead of 2?

(a) Let $a \in I^d$ and $b \in I^e$. Then given $a' \in I^{d+1}$ and $b' \in I^{e+1}$, we have $(a + a')(b + b') = ab + a'b + ab' + a'b' \in ab + I^{d+e+1}$.
(b) Note that $\operatorname{gr}_I(R)_d$ is exactly the vector space fT^d with $f \in R_d$. So "ignoring" T is an isomorphism of vector spaces. One checks directly that it is compatible with multiplication by reducing to the case of homogeneous elements.
(c) For example, if $f = X - 1$ and $g = 1$, then $f^* = -1$, $g^* = 1$, but $(f + g)^* = X$.
(d) Note that $\operatorname{gr}_I(R)_d$ is again just the vector space fT^d with $f \in R_d$, and multiplication is the same as in the polynomial case.
(e) The same thing.

(3) Consider the special case of Artin-Rees where $M = R$, and $I = (f)$ and $N = (g)$.
(a) What does Artin-Rees say in this setting? Express your answer in terms of "divides".
(b) Take $R = \mathbb{Z}$. Does $c = 0$ "work" for every $f, g \in \mathbb{Z}$? Can you find a sequence of examples requiring arbitrarily large values of c?

(a) There is some c such that $f^n|h$ and $g|h$ implies $(f^{n-c}g)|h$.
(b) Take $f = 2$ and $g = 2^m$. Then $2^n|h$ and $2^m|h$ implies $2^{\max\{m,n\}}|h$. Then $f^{n-c}g = 2^{m+n-c}$. To guarantee this to divide h, we must have $c \geq m$.

(4) Proof of Artin-Rees: Let R be a Noetherian ring, and I be an ideal.
(a) Explain why $R[It]$ is a Noetherian ring.
(b) Let $M = \sum_i Rm_i$ be a finitely generated R-module. Set $\mathcal{M} := \bigoplus_{n\geq 0} I^n M t^n$. Show that this is a graded $R[It]$-module, and that $\mathcal{M} = \sum_i R[It] \cdot m_i$, where in the last equality we consider m_i as the element $m_i t^0 \in \mathcal{M}_0$.
(c) Given a submodule N of M, set $\mathcal{N} := \bigoplus_{n\geq 0}(I^n M \cap N)t^n \subseteq \mathcal{M}$. Show that $\mathcal{N}$ is a graded $R[It]$-submodule of $\mathcal{M}$.
(d) Show that there exist $n_1, \dots, n_k \in N$ and $c_1, \dots, c_k \geq 0$ such that $\mathcal{N} = \sum_j R[It] \cdot n_j t^{c_j}$.
(e) Show that $c := \max\{c_j\}$ satisfies the conclusion of the Artin-Rees Lemma.

(a) Since I is finitely generated, it is a finitely generated algebra over a Noetherian ring.
(b) First, we check that this is an $R[It]$-module. It is clearly an additive group. To check that it is closed under the $R[It]$-action and that this yields a graded action, it suffices to check that $R[It]_d \cdot \mathcal{M}_e \subseteq \mathcal{M}_{d+e}$. To see it, take rt^d with $r \in I^d$ and mt^e with $m \in I^e M$; then the action yields rmt^{d+e} and $rm \in I^d(I^e M) = I^{d+e}M$, so $rmt^{d+e} \in \mathcal{M}_{d+e}$, as required.
Clearly $m_i \in \mathcal{M}$, so $\sum_i R[It] \cdot m_i \subseteq \mathcal{M}$. Now we check that this generates. It suffices to check that any homogeneous element can be generated by this generating set, so take some $mt^d \in \mathcal{M}_d$ with $m \in I^d M$. This means we can write $m = \sum_j a_j u_j$ with $a_j \in I^d$ and $u_j \in M$. Then we can write $u_j = \sum b_{ij} m_i$ for some $b_{ij} \in R$, yielding an expression $m = \sum_i c_i m_i$ with $c_i \in I^d$. Thus, $m = \sum_i (c_i t^d) m_i \in R[It] \cdot m_i$.
(c) It suffices to check that $R[It]_d \cdot \mathcal{N}_e \subseteq \mathcal{N}_{d+e}$. Take rt^d with $r \in I^d$ and nt^e with $n \in (I^e M \cap N)$. Then $rn \in I^d(I^e M \cap N)$, so $rn \in I^d I^e M = I^{d+e}M$ and $rn \in I^d N \subseteq N$, and hence $rn \in I^{d+e}M \cap N$. Thus $(rt^d)(nt^e) \in \mathcal{N}_{d+e}$.

(d) Since $R[It]$ is Noetherian and $\mathcal{M}$ is finitely generated, so is $\mathcal{N}$. Since it is graded and finitely generated, it can be generated by finitely many homogeneous elements. The statement is just naming them.

(e) Let $c = \max\{c_j\}$. Take $u \in I^n M \cap N$. Then $ut^n \in \mathcal{N}_n = \sum_j R[It] \cdot n_j t^{c_j}$. We can then express u as a homogeneous linear combination of these generators, so $ut^n = \sum_j (r_j t^{n-c_j})(n_j t^{c_j})$. Since $n - c_j \geq n - c$, we have $r_j \in I^{n-c}$, and each $n_j \in N$, so $u = \sum_j r_j n_j \in I^{n-c}N$. Moving over the c, we obtain the statement.

(5) Presentations of associated graded rings: Let R be a ring and I, J be ideals. Set $\mathrm{in}_I(J)$ to be the ideal of $\mathrm{gr}_I(R)$ generated by $\{a^* \mid a \in J\}$.

(a) Show that $\mathrm{gr}_I(R/J) \cong \mathrm{gr}_I(R)/\mathrm{in}(J)$.

(b) If $J = (f)$ is a principal ideal, show that $\mathrm{in}_I(J) = (f^*)$.

(c) Is $\mathrm{in}_I((f_1, \dots, f_t)) = (f_1^*, \dots, f_t^*)$ in general?

(d) Compute $\mathrm{gr}_{(x,y,z)}(\frac{K[\![X,Y,Z]\!]}{(X^2+XY+Y^3+Z^7)})$.

(6) Properties of associated graded rings: Let R be a ring and I be an ideal such that $\bigcap_{n\geq 0} I^n = 0$.

(a) Show that if $\mathrm{gr}_I(R)$ is a domain, then so is R.

(b) Show that if $\mathrm{gr}_I(R)$ is reduced, then so is R.

(c) What about the converses of these statements?

(7) Show that for the ideal $I = (X, Y)^2$ in $R = K[X, Y]$, the Rees ring $R[It]$ has defining relations of degree greater than one.