## §3.14: REES RINGS AND THE ARTIN-REES LEMMA

DEFINITION: Let $R$ be a ring and $I$ be an ideal. The **Rees ring** of $I$ is the $\mathbb{N}$-graded $R$-algebra

$$R[IT] := \bigoplus_{d\geq 0} I^d T^d = R \oplus IT \oplus I^2T^2 \oplus \cdots$$

with multiplication determined by $(aT^d)(bT^e) = abT^{d+e}$ for $a \in I^d$, $b \in I^e$ (and extended by the distributive law for nonhomogeneous elements). Here $I^n$ means the $n$th power of the ideal $I$ in $R$, and $T$ is an indeterminate. Equivalently, $R[IT]$ is the $R$-subalgebra of the polynomial ring $R[T]$ generated by $IT$, with $R[T]$ is given the standard grading $R[T]_d = R \cdot T^d$.

DEFINITION: Let $R$ be a ring and $I$ be an ideal. The **associated graded ring** of $I$ is the $\mathbb{N}$-graded ring

$$\mathrm{gr}_I(R) := \bigoplus_{d\geq 0} (I^d/I^{d+1})T^d = R/I \oplus (I/I^2)T \oplus (I^2/I^3)T^2 \oplus \cdots$$

with multiplication determined by $(a + I^{d+1}T^d)(b + I^{e+1}T^e) = ab + I^{d+e+1}\, T^{d+e}$ for $a \in I^d$, $b \in I^e$ (and extended by the distributive law). For an element $r \in R$, its **initial form** in $\mathrm{gr}_I(R)$ is

$$r^* := \begin{cases} (r + I^{d+1})T^d & \text{if } r \in I^d \smallsetminus I^{d+1} \\ 0 & \text{if } r \in \bigcap_{n\geq 0} I^n. \end{cases}$$

ARTIN-REES LEMMA: Let $R$ be a Noetherian ring, $I$ an ideal of $R$, $M$ a finitely generated module, and $N \subseteq M$ a submodule. Then there is a constant[1] $c \geq 0$ such that for all $n \geq c$, we have $I^n M \cap N \subseteq I^{n-c} N$.

**(1)** Warmup with Rees rings:
- **(a)** Let $R$ be a ring and $I$ be an ideal. Show that if $I = (a_1, \dots, a_n)$, then $R[IT] = R[a_1T, \dots, a_nT]$.
- **(b)** Let $K$ be a field, $R = K[X,Y]$ and $I = (X,Y)$. Find $K$-algebra generators for $R[IT]$, and find a relation on these generators.

**(2)** Warmup with associated graded rings:
- **(a)** Convince yourself that the multiplication given in the definition of $\mathrm{gr}_I(R)$ is well-defined. After doing this, do *not* use coset notation for elements of $\mathrm{gr}_I(R)$ and instead write a typical homogeneous element as something like $\overline{r}\, T^d$.
- **(b)** Let $K$ be a field, $R = K[X,Y]$, and $\mathfrak{m} = (X,Y)$. Show that $\mathrm{gr}_{\mathfrak{m}}(R)_d \cong R_d$ as $K$-vector spaces, and construct a ring isomorphism $\mathrm{gr}_{\mathfrak{m}}(R) \cong R$.
- **(c)** For the same $R$, show that the map $R \to \mathrm{gr}_{\mathfrak{m}}(R)$ given by $r \mapsto r^*$ is *not* a ring homomorphism.
- **(d)** Let $K$ be a field, $R = K[\![X,Y]\!]$, and $\mathfrak{m} = (X,Y)$. Show[2] that $\mathrm{gr}_{\mathfrak{m}}(R) \cong K[X,Y]$.
- **(e)** What happens in (b) and (d) if we have $n$ variables instead of 2?

**(3)** Consider the special case of Artin-Rees where $M = R$, and $I = (f)$ and $N = (g)$.
- **(a)** What does Artin-Rees say in this setting? Express your answer in terms of "divides".
- **(b)** Take $R = \mathbb{Z}$. Does $c = 0$ "work" for every $f, g \in \mathbb{Z}$? Can you find a sequence of examples requiring arbitrarily large values of $c$?

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[1] The constant $c$ depends on $I$, $M$, and $N$ but works for all $n$.

[2] Yes, the brackets changed. This is not a typo!

(4) Proof of Artin-Rees: Let $R$ be a Noetherian ring, and $I$ be an ideal.
  (a) Explain why $R[IT]$ is a Noetherian ring.
  (b) Let $M = \sum_i Rm_i$ be a finitely generated $R$-module. Set $\mathcal{M} := \bigoplus_{n\geq 0} I^n MT^n$. Show that this is a graded $R[IT]$-module, and that $\mathcal{M} = \sum_i R[IT] \cdot m_i$, where in the last equality we consider $m_i$ as the element $m_i T^0 \in \mathcal{M}_0$.
  (c) Given a submodule $N$ of $M$, set $\mathcal{N} := \bigoplus_{n\geq 0}(I^n M \cap N)T^n \subseteq \mathcal{M}$. Show that $\mathcal{N}$ is a graded $R[IT]$-submodule of $\mathcal{M}$.
  (d) Show that there exist $n_1, \dots, n_k \in N$ and $c_1, \dots, c_k \geq 0$ such that $\mathcal{N} = \sum_j R[It] \cdot n_j T^{c_j}$.
  (e) Show that $c := \max\{c_j\}$ satisfies the conclusion of the Artin-Rees Lemma.

(5) Presentations of associated graded rings: Let $R$ be a ring and $I, J$ be ideals. Set $\mathrm{in}_I(J)$ to be the ideal of $\mathrm{gr}_I(R)$ generated by $\{a^* \mid a \in J\}$.
  (a) Show that $\mathrm{gr}_I(R/J) \cong \mathrm{gr}_I(R)/\mathrm{in}(J)$.
  (b) If $J = (f)$ is a principal ideal, show that $\mathrm{in}_I(J) = (f^*)$.
  (c) Is $\mathrm{in}_I((f_1, \dots, f_t)) = (f_1^*, \dots, f_t^*)$ in general?
  (d) Compute $\mathrm{gr}_{(x,y,z)}\left(\dfrac{K[\![X,Y,Z]\!]}{(X^2+XY+Y^3+Z^7)}\right)$.

(6) Properties of associated graded rings: Let $R$ be a ring and $I$ be an ideal such that $\bigcap_{n\geq 0} I^n = 0$.
  (a) Show that if $\mathrm{gr}_I(R)$ is a domain, then so is $R$.
  (b) Show that if $\mathrm{gr}_I(R)$ is reduced, then so is $R$.
  (c) What about the converses of these statements?

(7) Show that for the ideal $I = (X,Y)^2$ in $R = K[X,Y]$, the Rees ring $R[IT]$ has defining relations of degree greater than one.