HILBERT'S FINITENESS THEOREM: Let K be a field of characteristic zero, and $R = K[X_1, \dots, X_n]$ be a polynomial ring. Let G be a finite group acting on R by degree-preserving K-algebra automorphisms. Then the invariant ring R^G is algebra-finite over K.

THEOREM: Let R be an $\mathbb{N}$-graded ring. Then R is Noetherian if and only if R_0 is Noetherian and R is algebra-finite over R_0.

DEFINITION: Let $R \subseteq S$ be an inclusion of rings. We say that R is a **direct summand** of S if there is an R-module homomorphism $\pi : S \to R$ such that $\pi|_R = \mathbb{1}_R$.

PROPOSITION: A direct summand of a Noetherian ring is Noetherian.

LEMMA: Let R be a polynomial ring over a field K. If G is a group acting on R by degree-preserving K-algebra automorphisms, then

(1) R^G is an $\mathbb{N}$-graded K-subalgebra of R with $(R^G)_0 = K$.
(2) If in addition, G is finite, and $|G|$ is invertible in K, then R^G is a direct summand of R.

(1) Use the Lemma, Proposition, and Theorem to deduce Hilbert's finiteness Theorem.

By the Lemma, R^G is a direct summand of R. Since R is Noetherian, so is R^G. By the Lemma, R^G is graded with $(R^G)_0 = K$. Then, by the Theorem, since R^G is Noetherian, and R^G is algebra-finite over $(R^G)_0$, and it remains to note that $(R^G)_0 = K$.

(2) Proof of Theorem:

(a) Explain the direction ($\Leftarrow$).
(b) Show that R Noetherian implies R_0 is Noetherian.
(c) Let $f_1, \dots, f_t$ be a homogeneous generating set for R_+, the ideal generated by positive degree elements of R. Show[1] by (strong) induction on d that every element of R_d is contained in $R_0[f_1, \dots, f_t]$.
(d) Conclude the proof of the Theorem.

(a) This follows from the Hilbert Basis Theorem.
(b) $R_0 \cong R/R_+$.
(c) For $d = 0$ there is nothing to show. For $d > 0$, take $h \in R_d$. Since $R_d \subseteq R_+$, write $h = \sum_i r_i f_i$ for some $r_i \in R$. If we replace r_i by r_i' its homogeneous component of degree $d - \deg(f_i)$, we claim that $h = \sum_i r_i' f_i$. Indeed, writing each r_i as a sum of homogeneous components and multiplying out, all of the other terms are homogeneous of some other degree, so the claim follows by uniqueness of homogeneous decomposition. So suppose r_i is homogeneous of degree $d - \deg(f_i)$. By induction, we have $r_i \in R_0[f_1, \dots, f_t]$. But then this plus $h = \sum_i r_i f_i$ show $h \in R_0[f_1, \dots, f_t]$.
(d) If R is Noetherian then R_+ is finitely generated as an ideal; since R_+ is homogeneous, it is generated by the (fintely many) components of these generators so has a finite homogeneous generating set, and a such generating set of R_+ generates R as an algebra over R_0 by the previous part.

[1]Hint: Start by writing $h \in R_d$ as $h = \sum_i r_i f_i$ with $d = \deg(r_i) + \deg(f_i)$ for all i.

(3) Proof of Proposition:

(a) Show that if R is a direct summand of S, and I is an ideal of R, then $IS \cap R = I$.

(b) Complete the proof of the proposition.

(a) We always have $I \subseteq IS \cap R$. Let $f \in IS \cap R$, so $f = \sum_i a_i s_i$ with $a_i \in I$, $s_i \in S$. Apply the map π. Since $f \in R$, we have $\pi(f) = f$. Since π is R-linear, we also have $\pi\left(\sum_i a_i s_i\right) = \sum_i a_i \pi(s_i)$, with $\pi(s_i) \in R$. But this is an element of I, so $f \in I$.

(b) Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be a chain of ideals in R. Then $I_1 S \subseteq I_2 S \subseteq I_3 S \subseteq \cdots$ is a chain of ideals in S, which necessarily stabilizes. But the chain $(I_1 S \cap R) \subseteq (I_2 S \cap R) \subseteq (I_3 S \cap R) \subseteq \cdots$ stabilizes, but this is our original chain!

(4) Proof of Lemma part (2): Consider $r \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot r$.

One checks directly that this map is R^G-linear and restricts to the identity on R^G.

(5) Show that a direct summand of a normal ring is normal.

(6) Let $\mathcal{S}_3$ denote the symmetric group on 3 letters, and let $\mathcal{S}_3$ act on $R = \mathbb{C}[X_1, X_2, X_3]$ by permuting variables; i.e., σ is the $\mathbb{C}$-algebra homomorphism given by $\sigma \cdot X_i = X_{\sigma(i)}$. Find a $\mathbb{C}$-algebra generating set for $R^{\mathcal{S}_3}$. What about replacing 3 by n?