DEFINITION: Let R be an $\mathbb{N}$-graded ring with graded pieces R_i. A **$\mathbb{Z}$-grading** on an R-module M is

- a decomposition of M as additive groups $M = \bigoplus_{e\in\mathbb{Z}} M_e$
- such that $r \in R_d$ and $m \in M_e$ implies $rm \in M_{d+e}$.

An **$\mathbb{Z}$-graded module** is a module with a $\mathbb{Z}$-grading. As with rings, we have the notions of **homogeneous** elements of M, the **degree** of a homogeneous element, **homogeneous decomposition** of an arbitrary element of M. A homomorphism $\phi : M \to N$ between graded modules is **degree-preserving** if $\phi(M_e) \subseteq N_e$.

GRADED NAK 1: Let R be an $\mathbb{N}$-graded ring, and R_+ be the ideal generated by the homogeneous elements of positive degree. Let M be a $\mathbb{Z}$-graded module. Suppose that $M_{\ll 0} = 0$; that is, there is some $n \in \mathbb{Z}$ such that $M_t = 0$ for $t \leq n$. Then $M = R_+M$ implies $M = 0$.

GRADED NAK 2: Let R be an $\mathbb{N}$-graded ring and M be a $\mathbb{Z}$-graded module with $M_{\ll 0} = 0$. Let N be a graded submodule of M. Then $M = N + R_+M$ if and only if $M = N$.

GRADED NAK 3: Let R be an $\mathbb{N}$-graded ring and M be a $\mathbb{Z}$-graded module with $M_{\ll 0} = 0$. Then a set of homogeneous elements $S \subseteq M$ generates M if and only if the image of S in M/R_+M generates M/R_+M as a module over $R_0 \cong R/R_+$.

DEFINITION: Let R be an $\mathbb{N}$-graded ring with $R_0 = K$ a field. Let M be a a $\mathbb{Z}$-graded module with $M_{\ll 0} = 0$. A set S of homogeneous elements of M is a **minimal generating set** for M if the image of S in M/R_+M is an K-vector space basis.

(1) Warmup with minimal generating sets.

(a) Note that the definition of "minimal generating set" does not say that it is a generating set. Use Graded NAK 3 to explain why it is!

(b) Let K be a field and $S = K[X, Y]$. Verify that $\{X^2, XY, Y^2\}$ is a minimal generating set of the ideal I it generates in S.

(c) Let K be a field. Find a minimal generating set of $S = K[X, Y]$ as a module over the K-subalgebra $R = K[X + Y, XY]$.

(a) A basis is a generating set; it is then the $(\Leftarrow)$ of Graded NAK 3.

(b) We need to show that the images of X^2, XY, Y^2 form a basis for I/R_+I; write lowercase for images in this quotient. To see that they span, take $F \in I$, so $F = AX^2 + BXY + CY^2$ for $A, B, C \in R$; then going modulo R_+ we have $f = ax^2+bxy+cy^2$, so x^2, xy, y^2 span the quotient. For linear independence, $ax^2+bxy+cy^2 = 0$ implies $AX^2+BXY+CY^2 \in R_+I$, and by comparing degrees, A, B, C have bottom degree one, hence are in R_+, so $a, b, c = 0$. Alternatively, note that I consists of all polynomials of bottom degree at least two, and R_+I consists of all polynomials of bottom degree at least three. Then the quotient is isomorphic as a vector space to the collection of polynomials of degree two, and X^2, XY, Y^2 is indeed a basis.

(c) We compute $S/R_+S = K[X, Y]/(X + Y, XY) \cong K[Y]/(-Y^2) \cong K[Y]/(Y^2)$, so the classes of $1, Y$ generate. Thus $\{1, Y\}$ forms a minimal generating set.

(2) Proofs of graded NAKs:

(a) Prove Graded NAK 1.

(b) Use Graded NAK 1 to prove Graded NAK 2.
(c) Use Graded NAK 2 to prove Graded NAK 3.

> **(a)** Suppose that $M \neq 0$. Take a nonzero homogeneous element m of minimal degree d in M, which exists by the hypothesis. Then since $m \in R_+ M$, we can write $r = \sum_i r_i m_i$ with $r_i \in R_+$, so the bottom degree of r_i is at least one. Thus, we can take the top degree of m_i to be $< d$. But then each $m_i = 0$, so $m = 0$, a contradiction.
> **(b)** The $(\Leftarrow)$ direction is clear. For the other, we can apply Graded NAK 1 to M/N since it is graded and its degrees are bounded below. We have $\frac{M}{N} = \frac{N+R_+M}{N} = R_+ \frac{M}{N}$ so $M/N = 0$; i.e., $M = N$.
> **(c)** Apply Graded NAK 2 to the submodule $N = \sum_{s \in S} Rs$: to do so, we need to note that a submodule generated by homogeneous elements is a graded submodule, which follows along similar lines to the corresponding statement we showed for ideals.

(3) The hypotheses:
(a) Examine your proofs from the previous problem and verify that one direction (each) of Graded NAK 2 and Graded NAK 3 hold without assuming that R or M is graded.
(b) Let K be a field and $R = K[X]$ with the standard grading. Let $M = K[X]/(X-1)$. Analyze the hypotheses and conclusion of Graded NAK 1 for this example.
(c) Let K be a field and $R = K[X]$ with the standard grading. Let $M = K[X, X^{-1}]$. Analyze the hypotheses and conclusion of Graded NAK 1 for this example.
(d) Find counterexamples to Graded NAK 3 with M is not graded or not bounded below in degree.

> **(a)** The $(\Leftarrow)$ direction of Graded NAK 2 and the $(\Rightarrow)$ direction of Graded NAK 3 hold without assuming that R or M is graded.
> **(b)** M is not a graded module; any element is of the form $\overline{\lambda}$ for $\lambda \in K$; if such an element was homogeneous, then
> $$\deg(\overline{\lambda}) = \deg(\overline{X\lambda}) = \deg(X) + \deg(\overline{\lambda}) = 1 + \deg(\overline{\lambda}),$$
> a contradiction. We also have $M = (X)M = R_+M$.
> **(c)** M is graded, but not bounded below. We also have $M = (X)M = R_+M$.
> **(d)** For a cheap example, take either of the previous with $S = \varnothing$.

(4) Minimal generating sets: Let R be an $\mathbb{N}$-graded ring with $R_0 = K$ a field. Let M be a a $\mathbb{Z}$-graded module with $M_{\ll 0} = 0$.
(a) Explain why every minimal generating set for M has the same cardinality.
(b) Explain why every homogeneous generating set for M contains a minimal generating set for M. Moreover, explain why any generating set (homogeneous or not) has cardinality at least that of a minimal generating set.
(c) Explain why "minimal generating set" is equivalent to "homogeneous generating set such that no proper subset generates".
(d) Give an example of a finitely generated module N over $K[X, Y]$ and two generating set S_1, S_2 for N such that no proper subset of S_i generates N, but $|S_1| \neq |S_2|$. Compare to the statements above.

> (a) Because all bases of a vector space do.
> (b) If S is a homogeneous generating set for M, then the images span M/R_+M, so the images must contain a basis; the elements of S that map to a basis form a minimal generating set. For a general generating set, its images still contain a basis of M/R_+M.

(c) This just follows from the fact that a basis of a vector space is the same as a minimal spanning set.

(d) One could take the two generating sets of the ideal $I = ((X-1)Y, XY) = (Y)$.

(5) Let R be an $\mathbb{N}$-graded ring with $R_0 = K$ a field. Suppose that $R_{\text{red}} = R/\sqrt{0}$ is a domain, and that $f \in R$ is a homogeneous nonnilpotent element of positive degree. Show that $R/(f)$ is reduced implies that R is a reduced, and hence a domain.

(6) Let $r \in \sqrt{0}$ be a homogeneous nilpotent element. Then for some $e \in \mathbb{N}$ we have $r^e = 0 \in (f)$, and since $R/(f)$ is reduced, $r \in (f)$. Thus, we can write $r = fs$ for some homogeneous s. But $r \in \sqrt{0}$, $f \notin \sqrt{0}$, and $\sqrt{0}$ prime implies that $s \in \sqrt{0}$. This implies that $\sqrt{0} = f\sqrt{0} \subseteq R_+\sqrt{0}$, so $\sqrt{0} = 0$; i.e., R is reduced.