§3.12: GRADED MODULES

DEFINITION: Let R be an $\mathbb{N}$-graded ring with graded pieces R_i. A **$\mathbb{Z}$-grading** on an R-module M is

- a decomposition of M as additive groups $M = \bigoplus_{e\in\mathbb{Z}} M_e$
- such that $r \in R_d$ and $m \in M_e$ implies $rm \in M_{d+e}$.

An **$\mathbb{Z}$-graded module** is a module with a $\mathbb{Z}$-grading. As with rings, we have the notions of **homogeneous** elements of M, the **degree** of a homogeneous element, **homogeneous decomposition** of an arbitrary element of M. A homomorphism $\phi : M \to N$ between graded modules is **degree-preserving** if $\phi(M_e) \subseteq N_e$.

GRADED NAK 1: Let R be an $\mathbb{N}$-graded ring, and R_+ be the ideal generated by the homogeneous elements of positive degree. Let M be a $\mathbb{Z}$-graded module. Suppose that $M_{\ll 0} = 0$; that is, there is some $n \in \mathbb{Z}$ such that $M_t = 0$ for $t \leq n$. Then $M = R_+M$ implies $M = 0$.

GRADED NAK 2: Let R be an $\mathbb{N}$-graded ring and M be a $\mathbb{Z}$-graded module with $M_{\ll 0} = 0$. Let N be a graded submodule of M. Then $M = N + R_+M$ if and only if $M = N$.

GRADED NAK 3: Let R be an $\mathbb{N}$-graded ring and M be a $\mathbb{Z}$-graded module with $M_{\ll 0} = 0$. Then a set of homogeneous elements $S \subseteq M$ generates M if and only if the image of S in M/R_+M generates M/R_+M as a module over $R_0 \cong R/R_+$.

DEFINITION: Let R be an $\mathbb{N}$-graded ring with $R_0 = K$ a field. Let M be a a $\mathbb{Z}$-graded module with $M_{\ll 0} = 0$. A set S of homogeneous elements of M is a **minimal generating set** for M if the image of S in M/R_+M is an K-vector space basis.

(1) Warmup with minimal generating sets.
- **(a)** Note that the definition of "minimal generating set" does not say that it is a generating set. Use Graded NAK 3 to explain why it is!
- **(b)** Let K be a field and $S = K[X, Y]$. Verify that $\{X^2, XY, Y^2\}$ is a minimal generating set of the ideal I it generates in S.
- **(c)** Let K be a field. Find a minimal generating set of $S = K[X, Y]$ as a module over the K-subalgebra $R = K[X + Y, XY]$.

(2) Proofs of graded NAKs:
- **(a)** Prove Graded NAK 1.
- **(b)** Use Graded NAK 1 to prove Graded NAK 2.
- **(c)** Use Graded NAK 2 to prove Graded NAK 3.

(3) The hypotheses:
- **(a)** Examine your proofs from the previous problem and verify that one direction (each) of Graded NAK 2 and Graded NAK 3 hold without assuming that R or M is graded.
- **(b)** Let K be a field and $R = K[X]$ with the standard grading. Let $M = K[X]/(X - 1)$. Analyze the hypotheses and conclusion of Graded NAK 1 for this example.
- **(c)** Let K be a field and $R = K[X]$ with the standard grading. Let $M = K[X, X^{-1}]$. Analyze the hypotheses and conclusion of Graded NAK 1 for this example.
- **(d)** Find counterexamples to Graded NAK 3 with M is not graded or not bounded below in degree.

(4) Minimal generating sets: Let R be an $\mathbb{N}$-graded ring with $R_0 = K$ a field. Let M be a a $\mathbb{Z}$-graded module with $M_{\ll 0} = 0$.
   (a) Explain why every minimal generating set for M has the same cardinality.
   (b) Explain why every homogeneous generating set for M contains a minimal generating set for M. Moreover, explain why any generating set (homogeneous or not) has cardinality at least that of a minimal generating set.
   (c) Explain why "minimal generating set" is equivalent to "homogeneous generating set such that no proper subset generates".
   (d) Give an example of a finitely generated module N over $K[X, Y]$ and two generating set S_1, S_2 for N such that no proper subset of S_i generates N, but $|S_1| \neq |S_2|$. Compare to the statements above.

(5) Let R be an $\mathbb{N}$-graded ring with $R_0 = K$ a field. Suppose that $R_{\text{red}} = R/\sqrt{0}$ is a domain, and that $f \in R$ is a homogeneous nonnilpotent element of positive degree. Show that $R/(f)$ is reduced implies that R is a reduced, and hence a domain.