DEFINITION:

(1) An **$\mathbb{N}$-grading** on a ring R is
- a decomposition of R as additive groups $R = \bigoplus_{d \geq 0} R_d$
- such that $x \in R_d$ and $y \in R_e$ implies $xy \in R_{d+e}$.

(2) An **$\mathbb{N}$-graded ring** is a ring with an $\mathbb{N}$-grading.

(3) We say that an element $x \in R$ in an $\mathbb{N}$-graded ring R is **homogeneous** of **degree** d if $x \in R_d$.

(4) The **homogeneous decomposition** of an element $r \neq 0$ in an $\mathbb{N}$-graded ring is the sum

$$r = r_{d_1} + \cdots + r_{d_k} \quad \text{where } r_{d_i} \neq 0 \text{ homogeneous of degree } d_i \text{ and } d_1 < \cdots < d_k.$$

The element r_{d_i} is the **homogeneous component** r **of degree** d_i.

(5) An ideal I in an $\mathbb{N}$-graded ring is **homogeneous** if $r \in I$ implies every homogenous component of r is in I. Equivalently, I is homogeneous if can be generated by homogeneous elements.

(6) A homomorphism $\phi : R \to S$ between $\mathbb{N}$-graded rings is **graded** if $\phi(R_d) \subseteq S_d$ for all $d \in \mathbb{N}$.

DEFINITION: For an abelian semigroup $(G, +)$, one defines G**-grading** as above with G in place of $\mathbb{N}$ and $g \in G$ in place of $d \geq 0$. The other definitions above make sense in this context.

DEFINITION: Let K be a field, and $R = K[X_1, \dots, X_n]$ be a polynomial ring. Let G be a group acting on R so that for every $g \in G$, $r \mapsto g \cdot r$ is a K-algebra homomorphism. The **ring of invariants** of G is

$$R^G := \{r \in R \mid \text{for all } g \in G,\ g \cdot r = r\}.$$

(1) Basics with graded rings: Let R be an $\mathbb{N}$-graded ring.

(a) If $f \in R$ is homogeneous of degree a and $g \in R$ is homogeneous of degree b, what about $f+g$ and fg?

(b) Translate the definition of graded ring to explain why every nonzero element has a unique homogeneous decomposition.

(c) Does every element in R have a degree? What about "top degree" or "bottom degree"?

(d) What is the[1] degree of zero?

(e) Suppose that $r \in (s_1, \dots, s_m)$, and r is homogeneous of degree d, and s_i is homogeneous of degree d_i. Explain why we can write $r = \sum_i a_i s_i$ with $a_i \in R$ *homogeneous* of degree $d - d_i$.

(a) $f+g$ is homogeneous if and only if $a = b$, in which case it has degree a; fg is homogeneous of degree $a+b$.

(b) The direct sum decomposition means that every element can be expressed in a unique way as a finite sum of elements from the components.

(c) No; only homogeneous elements have a degree. Any nonzero element has a top degree and a bottom degree.

(d) Zero is homogeneous of every degree, since each R_n is an additive group.

(e) We can write $r = \sum_i b_i s_i$ for some $b_i \in R$. Write $b_i = a_i + c_i$ where a_i is the homogeneous component of degree $d - d_i$ (or zero, if there is none) and c_i is the sum of the other components. Then $r = \sum_i a_i s_i + \sum_i c_i s_i$ where $\sum_i a_i s_i$ has degree d and $\sum_i c_i s_i$ lives entirely in other degrees. By comparing homogeneous components, we must have $\sum_i a_i s_i = r$ (and $\sum_i c_i s_i = 0$).

[1] Hint: This is a trick question, but specify exactly how.

(2) The **standard grading** on a polynomial ring: Let A be a ring.

(a) Let $R = A[X]$. Discuss: the decomposition $R_d = A \cdot X^d$ gives an $\mathbb{N}$-grading on R.

(b) Let $R = A[X_1, \dots, X_n]$. Discuss: the decomposition

$$R_d = \sum_{d_1+\cdots+d_n=d} A \cdot X_1^{d_1} \cdots X_m^{d_m}$$

gives an $\mathbb{N}$-grading on R. What is the homogeneous decomposition of $f = X_1^3 + 2X_1X_2 - X_3^2 + 3$?

(c) Let $R = A[\![X]\!]$. Explain why $R_n = A \cdot X^n$ *does not* give an $\mathbb{N}$-grading on R.

> **(a)** Agree.
> **(b)** Agree. $f_3 = X_1^3$, $f_2 = 2x_1x_2 - x_3^2$, $f_0 = 3$.
> **(c)** An element must be a finite sum of homogeneous elements.

(3) **Weighted gradings** on polynomial rings: Let A be a ring, $R = A[X_1, \dots, X_n]$ and $a_1, \dots, a_m \in \mathbb{N}$.

(a) Discuss: $R_n = \sum_{d_1a_1+\cdots+d_ma_m=n} A \cdot X_1^{d_1} \cdots X_m^{d_m}$ gives an $\mathbb{N}$-grading of R where the degree of X_i is a_i.

(b) Can you find a_1, a_2, a_3 such that $X_1^2 + X_2^3 + X_3^5$ is homogeneous? Of what degree?

> **(a)** Yes. It is the truth.
> **(b)** $a_1 = 15, a_2 = 10, a_3 = 6$ makes the element degree 30.

(4) The **fine grading** on polynomial rings: Let A be a ring and $R = A[X_1, \dots, X_n]$. Discuss why

$$R_d = A \cdot X^d \quad \text{for } d = (d_1, \dots, d_m) \in \mathbb{N}^n, \quad \text{where } \; X^d := X_1^{d_1} \cdots X_m^{d_m}$$

yields an $\mathbb{N}^m$-grading on R. What are the homogeneous elements?

> Yes, every polynomial is a sum of monomials with coefficients in a unique way, and the exponent vectors add when we multiply. The homogeneous elements are monomials with coefficients.

(5) More basics with graded rings. Let R be $\mathbb{N}$-graded.

(a) Show[2] that if $e \in R$ is idempotent, then e is homogeneous of degree zero. In particular, 1 is homogeneous of degree zero.

(b) Show that R_0 is a subring of R, and each R_n is an R_0-module.

(c) Show that if I is homogeneous, then R/I is also $\mathbb{N}$-graded where $(R/I)_n$ consists of the classes of homogeneous elements of R of degree n.

(d) Show that I is homogeneous if and only if I is generated by homogeneous elements.

(e) Suppose that $\phi : R \to S$ is a homomorphism of K-algebras, and that R and S are $\mathbb{N}$-graded with K contained in R_0 and S_0. Show that ϕ is graded if ϕ preserves degrees for all of the elements in some homogeneous generating set of R.

> (a) Suppose otherwise; then we can write $e = e_0 + e_d + X$ with e_0 the degree zero component (a priori possibly zero), $e_d \neq 0$ the lowest positive degree component, and X a sum of higher degree terms. Then $e^2 = e$ yields $e_0^2 + 2e_0e_d +$ higher degree terms $= e_0 + e_d +$ higher degree terms, and equating terms of the same degree, $e_0^2 = e_0$ and $2e_0e_d = e_d$. Multiplying the latter by e_0 and using the first gives $2e_0e_d = e_0e_d$, so $e_0e_d = 0$, so $e_d = 0$. This is a contradiction, so we must have $e = e_0$ is homogeneous of degree zero.

[2] Hint: If not, write $e = e_0 + e_d + X$ where e_0 has degree zero and e_d is the lowest nonzero positive degree component. Apply uniqueness of homogeneous decomposition to $e^2 = e$ and show that $2e_0e_d = e_0e_d \dots$

(b) From the above, $1 \in R_0$; we also know that R_0 is closed under $\pm$ and $\times$, so it is a subring. For $r \in R_0$ and $s \in R_n$, $rs \in R_n$, and all the other module axioms follows from the ring axioms in R.

(c) We need to show that R/I has a unique expression as a sum of elements in distinct $(R/I)_n$ pieces. Let $\overline{r} \in R/I$, and write $r = \sum_i r_{d_i}$ as a sum of homogeneous components. Then $\overline{r} = \sum_i \overline{r_{d_i}}$ gives existence. For uniqueness, suppose that $\overline{0} = \sum_i \sum_i \overline{r_{d_i}}$ with $r_{d_i} \in R_{d_i}$ and d_i distinct. This just means that $\sum_i r_{d_i} \in I$, and by definition of homogeneous ideal, we must have $r_{d_i} \in I$, so $\overline{r_{d_i}}$=$\overline{0}$. This is the required uniqueness statement.

(d) ($\Rightarrow$) Suppose that I is homogeneous, and let S be a generating set for I. We claim that the set of homogeneous components S' of elements of S is a generating set for I. Indeed, each such component is in I, so $(S') \subseteq I$ and since each generator is a linear combination of said components, we have $I = (S) \subseteq (S')$, so $(S') = I$. ($\Leftarrow$) Suppose that I is generated by a set S of homogeneous elements. Then given $f \in I$, we can write $f = \sum_i r_i s_i$ for some $s_i \in S$ of degree d_i. Write each r_i as a sum of homogeneous elements $r_i = \sum_j r_{i,j}$ with $\deg(r_{i,j}) = j$. Then $f = \sum_i r_i s_i = \sum_i \sum_j r_{i,j} s_i$. Then the homogeneous components of f are $\sum_{i,j:j+d_i=t} r_{i,j} s_i$, which lie in I.

(e) Any homogeneous element can be written as a polynomial expression in the generators: $r = \sum_i k_i f_1^{d_1} \cdots f_t^{d_t}$. Each summand on the right hand side is homogeneous, so taking the homogeneous component of degree equal to that of r, we can assume that each term in the right hand side had degree equal to that of r. Then $\phi(r) = \phi\left(\sum_i k_i f_1^{d_1} \cdots f_t^{d_t}\right) = \sum_i k_i \phi(f_1)^{d_1} \cdots \phi(f_t)^{d_t}$. But since $\deg(f_i) = \deg(\phi(f_i))$ the right hand side has the same degree as that on the previous formula, so $\deg(\phi(r)) = \deg(r)$.

(6) Semigroup rings: Let S be a subsemigroup of $\mathbb{N}^n$ with operation $+$ and identity $(0, \dots, 0)$. The **semigroup ring** of S is

$$K[S] := \sum_{\alpha \in S} K X^\alpha \subseteq R, \qquad \text{where } X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}.$$

(a) Show that $K[S]$ is a K-subalgebra that is a graded subring of R in the fine grading.
(b) Let $S = \langle 4, 7, 9 \rangle \subseteq \mathbb{N}$. Draw a picture of S. What is $K[S]$?
(c) Find a semigroup $S \subseteq \mathbb{N}^2$ such that $K[S]$ is Noetherian, and another such that $K[S]$ is not Noetherian. Draw pictures of these semigroups.
(d) Show that every K-subalgebra that is a graded subring of R in the fine grading is of the form $K[S]$ for some S.

(7) Homogeneous elements: Let R be an $\mathbb{N}$-graded ring.
(a) Show that R is a domain if and only if for all homogeneous elements x, y, $xy = 0$ implies $x = 0$ or $y = 0$.
(b) Show that the radical of a homogeneous ideal is homogeneous.

(8) In the setting of the definition of "ring of invariants" suppose that each $g \in G$ acts as a graded homomorphism. Show that R^G is an $\mathbb{N}$-graded K-subalgebra of R.