§3.11: Graded Rings

Definition:

(1) An $\mathbb{N}$**-grading** on a ring R is
- a decomposition of R as additive groups $R = \bigoplus_{d \geq 0} R_d$
- such that $x \in R_d$ and $y \in R_e$ implies $xy \in R_{d+e}$.

(2) An $\mathbb{N}$**-graded ring** is a ring with an $\mathbb{N}$-grading.

(3) We say that an element $x \in R$ in an $\mathbb{N}$-graded ring R is **homogeneous** of **degree** d if $x \in R_d$.

(4) The **homogeneous decomposition** of an element $r \neq 0$ in an $\mathbb{N}$-graded ring is the sum

$$r = r_{d_1} + \cdots + r_{d_k} \quad \text{where } r_{d_i} \neq 0 \text{ homogeneous of degree } d_i \text{ and } d_1 < \cdots < d_k.$$

The element r_{d_i} is the **homogeneous component** r **of degree** d_i.

(5) An ideal I in an $\mathbb{N}$-graded ring is **homogeneous** if $r \in I$ implies every homogenous component of r is in I. Equivalently, I is homogeneous if can be generated by homogeneous elements.

(6) A homomorphism $\phi : R \to S$ between $\mathbb{N}$-graded rings is **graded** if $\phi(R_d) \subseteq S_d$ for all $d \in \mathbb{N}$.

Definition: For an abelian semigroup $(G, +)$, one defines G**-grading** as above with G in place of $\mathbb{N}$ and $g \in G$ in place of $d \geq 0$. The other definitions above make sense in this context.

Definition: Let K be a field, and $R = K[X_1, \dots, X_n]$ be a polynomial ring. Let G be a group acting on R so that for every $g \in G$, $r \mapsto g \cdot r$ is a K-algebra homomorphism. The **ring of invariants** of G is

$$R^G := \{r \in R \mid \text{for all } g \in G,\ g \cdot r = r\}.$$

(1) Basics with graded rings: Let R be an $\mathbb{N}$-graded ring.
- **(a)** If $f \in R$ is homogeneous of degree a and $g \in R$ is homogeneous of degree b, what about $f + g$ and fg?
- **(b)** Translate the definition of graded ring to explain why every nonzero element has a unique homogeneous decomposition.
- **(c)** Does every element in R have a degree? What about "top degree" or "bottom degree"?
- **(d)** What is the[1] degree of zero?
- **(e)** Suppose that $r \in (s_1, \dots, s_m)$, and r is homogeneous of degree d, and s_i is homogeneous of degree d_i. Explain why we can write $r = \sum_i a_i s_i$ with $a_i \in R$ *homogeneous* of degree $d - d_i$.

(2) The **standard grading** on a polynomial ring: Let A be a ring.
- **(a)** Let $R = A[X]$. Discuss: the decomposition $R_d = A \cdot X^d$ gives an $\mathbb{N}$-grading on R.
- **(b)** Let $R = A[X_1, \dots, X_n]$. Discuss: the decomposition

$$R_d = \sum_{d_1 + \cdots + d_n = d} A \cdot X_1^{d_1} \cdots X_m^{d_m}$$

gives an $\mathbb{N}$-grading on R. What is the homogeneous decomposition of $f = X_1^3 + 2X_1X_2 - X_3^2 + 3$?
- **(c)** Let $R = A[\![X]\!]$. Explain why $R_n = A \cdot X^n$ *does not* give an $\mathbb{N}$-grading on R.

(3) **Weighted gradings** on polynomial rings: Let A be a ring, $R = A[X_1, \dots, X_n]$ and $a_1, \dots, a_m \in \mathbb{N}$.
- **(a)** Discuss: $R_n = \sum_{d_1 a_1 + \cdots + d_m a_m = n} A \cdot X_1^{d_1} \cdots X_m^{d_m}$ gives an $\mathbb{N}$-grading of R where the degree of X_i is a_i.
- **(b)** Can you find a_1, a_2, a_3 such that $X_1^2 + X_2^3 + X_3^5$ is homogeneous? Of what degree?

[1] Hint: This is a trick question, but specify exactly how.

(4) The **fine grading** on polynomial rings: Let A be a ring and $R = A[X_1, \dots, X_n]$. Discuss why

$$R_d = A \cdot X^d \quad \text{for } d = (d_1, \dots, d_m) \in \mathbb{N}^n, \text{ where } X^d := X_1^{d_1} \cdots X_m^{d_m}$$

yields an $\mathbb{N}^m$-grading on R. What are the homogeneous elements?

(5) More basics with graded rings. Let R be $\mathbb{N}$-graded.
 (a) Show[2] that if $e \in R$ is idempotent, then e is homogeneous of degree zero. In particular, 1 is homogeneous of degree zero.
 (b) Show that R_0 is a subring of R, and each R_n is an R_0-module.
 (c) Show that if I is homogeneous, then R/I is also $\mathbb{N}$-graded where $(R/I)_n$ consists of the classes of homogeneous elements of R of degree n.
 (d) Show that I is homogeneous if and only if I is generated by homogeneous elements.
 (e) Suppose that $\phi : R \to S$ is a homomorphism of K-algebras, and that R and S are $\mathbb{N}$-graded with K contained in R_0 and S_0. Show that ϕ is graded if ϕ preserves degrees for all of the elements in some homogeneous generating set of R.

(6) Semigroup rings: Let S be a subsemigroup of $\mathbb{N}^n$ with operation $+$ and identity $(0, \dots, 0)$. The **semigroup ring** of S is

$$K[S] := \sum_{\alpha \in S} K X^\alpha \subseteq R, \qquad \text{where } X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}.$$

 (a) Show that $K[S]$ is a K-subalgebra that is a graded subring of R in the fine grading.
 (b) Let $S = \langle 4, 7, 9 \rangle \subseteq \mathbb{N}$. Draw a picture of S. What is $K[S]$?
 (c) Find a semigroup $S \subseteq \mathbb{N}^2$ such that $K[S]$ is Noetherian, and another such that $K[S]$ is not Noetherian. Draw pictures of these semigroups.
 (d) Show that every K-subalgebra that is a graded subring of R in the fine grading is of the form $K[S]$ for some S.

(7) Homogeneous elements: Let R be an $\mathbb{N}$-graded ring.
 (a) Show that R is a domain if and only if for all homogeneous elements x, y, $xy = 0$ implies $x = 0$ or $y = 0$.
 (b) Show that the radical of a homogeneous ideal is homogeneous.

(8) In the setting of the definition of "ring of invariants" suppose that each $g \in G$ acts as a graded homomorphism. Show that R^G is an $\mathbb{N}$-graded K-subalgebra of R.

[2]Hint: If not, write $e = e_0 + e_d + X$ where e_0 has degree zero and e_d is the lowest nonzero positive degree component. Apply uniqueness of homogeneous decomposition to $e^2 = e$ and show that $2e_0e_d = e_0e_d \dots$