DEFINITION: A ring R is **Noetherian** if every ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ eventually stabilizes: i.e., there is some N such that $I_n = I_N$ for all $n \geq N$.

HILBERT BASIS THEOREM: If R is a Noetherian ring, then the polynomial ring $R[X]$ and power series ring $R[\![X]\!]$ are also Noetherian.

We will return to the proof of Hilbert Basis Theorem after discussing Noetherian modules next time.

COROLLARY: Every finitely generated algebra over a field is Noetherian.

(1) Equivalences for Noetherianity.
- **(a)** Show[1] that R is Noetherian if and only if every ideal is finitely generated.
- **(b)** Show[2] that R is Noetherian if and only if every nonempty collection of ideals has a maximal[3] element.

(a) ($\Leftarrow$) Suppose that every ideal is finitely generated, and take a chain $I_1 \subseteq I_2 \subseteq \cdots$. Consider $I = \bigcup_n I_n$. This is an ideal (it was important that we had a chain, not an arbitrary collection of ideals for this step), and by hypothesis we have $I = (f_1, \dots, f_m)$. For each i, there is some n_i such that $f_i \in I_{n_i}$. Let $N = \max\{n_i\}$. Then $I = (f_1, \dots, f_m) \subseteq I_N \subseteq I$, so equality holds, and the chain stabilizes at N.
($\Rightarrow$) Suppose that there is an ideal I that is not finitely generated. Then we construct an infinite chain as follows: let $f_1 \in I \smallsetminus 0$ (0 is finitely generated so $I \neq 0$), and set $I_1 = (f_1)$, and for each n take $f_{n+1} \in I \smallsetminus I_n = (f_1, \dots, f_n)$, ($I_n$ is finitely generated so $I \neq I_n$).

(b) ($\Leftarrow$) Suppose that every nonempty collection of ideals has a maximal element. Then a chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ is, in particular, a nonempty collection of ideals, hence has a maximal element, say I_n. Then for $n \geq n$, $I_N \subseteq I_n$ and maximality of I_N imply $I_N = I_n$.
($\Rightarrow$) Suppose that there is a nonempty collection of ideals without a maximal element, say $\mathcal{S}$. Let I_1 be any element of $\mathcal{S}$. Then, by definition, there is some I_2 that properly contains I_1, and so on, yielding a chain that does not stabilize.

(2) Some Noetherian rings:
- **(a)** Show that fields and PIDs are Noetherian.
- **(b)** Show that if R is Noetherian and $I \subseteq R$, then R/I is Noetherian.
- **(c)** Is[4] every subring of a Noetherian ring Noetherian?

(a) Every element of a field is generated by no elements; every element of a PID is generated by one element.

(b) The ideals of R/I are in containment-preserving bijection with ideals of R containing I. A chain of ideals in R containing I must stabilize, so the corresponding chain in R/I must stabilize as well.

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[1]For the backward direction, consider $\bigcup_{n\in\mathbb{N}} I_n$

[2]Hint: For the forward direction, show the contrapositive.

[3]This means that if $\mathcal{S}$ is our collection of ideals, there is some $I \in \mathcal{S}$ such that no $J \in \mathcal{S}$ properly contains I. It does not mean that there is a maximal ideal in $\mathcal{S}$.

[4]Hint: Every domain has a fraction field, even the domain from (4a).

(c) No: $K[X_1, X_2, \dots]$ is not Noetherian, but it is a subring of its fraction field $K(X_1, X_2, \dots)$, which is a field, hence Noetherian.

(3) Use the Hilbert Basis Theorem to deduce the Corollary.

From the Hilbert Basis Theorem and induction, if R is Noetherian, then $R[X_1, \dots, X_n]$ is as well. In particular, if K is a field, then $K[X_1, \dots, X_n]$ is too. Since a finitely generated K-algebra is a quotient of some $K[X_1, \dots, X_n]$, then any such ring is Noetherian as well.

(4) Some nonNoetherian rings:

(a) Let K be a field. Show that $K[X_1, X_2, \dots]$ is not Noetherian.

(b) Let K be a field. Show that $K[X, XY, XY^2, \dots]$ is not Noetherian.

(c) Show that $\mathcal{C}([0,1], \mathbb{R})$ is not Noetherian.

(a) The ideal $(X_1, X_2, \dots)$ is not finitely generated.

(b) The ideal $(X, XY, \dots)$ is not finitely generated.

(c) The ideal $\sqrt{(x)} = \mathfrak{m}_0$ is not finitely generated.

(5) Let R be a Noetherian ring. Show that for every ideal I, there is some n such that $\sqrt{I}^n \subseteq I$. In particular, there is some n such that for every nilpotent element z, $z^n = 0$.

Let $\sqrt{I} = (f_1, \dots, f_m)$. For each i, there is some n_i such that $f_i^{n_i} \in I$. Then for $n \geq n_1 + \cdots + n_m - m + 1$, any generator $f_1^{a_1} \cdots f_m^{a_m}$ with $\sum a_i = n$ must have $a_j \geq n_j$ for some j, and hence $f_1^{a_1} \cdots f_m^{a_m} \in I$.

For the particular case, we consider $\sqrt{0}$.

(6) Let R be Noetherian. Show that every element of R admits a decomposition into irreducibles.

We argue the contrapositive. Suppose that $r \in R$ does not admit a decomposition into irreducibles. Then in particular, r is reducible, so $r = r_1 r_1'$, with r_1' not a unit, so $(r) \subsetneqq (r_1)$. Likewise, r_1 is reducible, so $r_1 = r_2 r_2'$, with r_2' not a unit, so $(r_1) \subsetneqq (r_2)$. We can continue like this forever to obtain an infinite ascending chain of *principal* ideals even.

(7) Prove the principle of **Noetherian induction**: Let $\mathcal{P}$ be a property of a ring. Suppose that "For every nonzero ideal I, $\mathcal{P}$ is true for R/I implies that $\mathcal{P}$ is true for R" and $\mathcal{P}$ holds for all fields. Then $\mathcal{P}$ is true for every Noetherian ring.

(8) (a) Suppose that every maximal ideal of R is finitely generated. Must R be Noetherian?

(b) Suppose that every ascending chain of prime ideals stabilizes. Must R be Noetherian?

(c) Suppose that every prime ideal of R is finitely generated. Must R be Noetherian?

(a) No. One counterexample is $\mathcal{C}^\infty([0,1], \mathbb{R})$. Prove it!

Here is another more algebraic example: Let K be a field, and R be the subring of $K(X,Y)$ consisting of elements that can be written as f/g with $f = aX^n + bY$ and $g = uX^n + cY$ for some $n \geq 0$, $a, b, c \in K[X,Y]$, and $u \in K[X,Y]$ with nonzero constant term. I leave it to you to show that

- R is indeed a subring of $K(X,Y)$,
- the ideal (X) is a maximal ideal,
- any $r \in R \setminus (X)$ is a unit, so (X) is the unique maximal ideal, and
- the ideal $(Y, Y/X, Y/X^2, \dots)$ is not finitely generated.

This example is not totally coming from nowhere; see if you can find the train of thought behind it.

(b) No.

(c) Yes.