# §2.9: NOETHERIAN RINGS

DEFINITION: A ring $R$ is **Noetherian** if every ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ eventually stabilizes: i.e., there is some $N$ such that $I_n = I_N$ for all $n \geq N$.

HILBERT BASIS THEOREM: If $R$ is a Noetherian ring, then the polynomial ring $R[X]$ and power series ring $R[\![X]\!]$ are also Noetherian.

We will return to the proof of Hilbert Basis Theorem after discussing Noetherian modules next time.

COROLLARY: Every finitely generated algebra over a field is Noetherian.

**(1)** Equivalences for Noetherianity.
- **(a)** Show[1] that $R$ is Noetherian if and only if every ideal is finitely generated.
- **(b)** Show[2] that $R$ is Noetherian if and only if every nonempty collection of ideals has a maximal[3] element.

**(2)** Some Noetherian rings:
- **(a)** Show that fields and PIDs are Noetherian.
- **(b)** Show that if $R$ is Noetherian and $I \subseteq R$, then $R/I$ is Noetherian.
- **(c)** Is[4] every subring of a Noetherian ring Noetherian?

**(3)** Use the Hilbert Basis Theorem to deduce the Corollary.

**(4)** Some nonNoetherian rings:
- **(a)** Let $K$ be a field. Show that $K[X_1, X_2, \ldots]$ is not Noetherian.
- (b) Let $K$ be a field. Show that $K[X, XY, XY^2, \ldots]$ is not Noetherian.
- (c) Show that $\mathcal{C}([0,1], \mathbb{R})$ is not Noetherian.

(5) Let $R$ be a Noetherian ring. Show that for every ideal $I$, there is some $n$ such that $\sqrt{I}^n \subseteq I$. In particular, there is some $n$ such that for every nilpotent element $z$, $z^n = 0$.

(6) Let $R$ be Noetherian. Show that every element of $R$ admits a decomposition into irreducibles.

(7) Prove the principle of **Noetherian induction**: Let $\mathcal{P}$ be a property of a ring. Suppose that "For every nonzero ideal $I$, $\mathcal{P}$ is true for $R/I$ implies that $\mathcal{P}$ is true for $R$" and $\mathcal{P}$ holds for all fields. Then $\mathcal{P}$ is true for every Noetherian ring.

(8)
- (a) Suppose that every maximal ideal of $R$ is finitely generated. Must $R$ be Noetherian?
- (b) Suppose that every ascending chain of prime ideals stabilizes. Must $R$ be Noetherian?
- (c) Suppose that every prime ideal of $R$ is finitely generated. Must $R$ be Noetherian?

---

[1] For the backward direction, consider $\bigcup_{n \in \mathbb{N}} I_n$

[2] Hint: For the forward direction, show the contrapositive.

[3] This means that if $\mathcal{S}$ is our collection of ideals, there is some $I \in \mathcal{S}$ such that no $J \in \mathcal{S}$ properly contains $I$. It does not mean that there is a maximal ideal in $\mathcal{S}$.

[4] Hint: Every domain has a fraction field, even the domain from (4a).