DEFINITION: Let R be a domain. The **normalzation** of R is the integral closure of R in $\mathrm{Frac}(R)$. We say that R is **normal** if it is equal to its normalization, i.e., if R is integrally closed in its fraction field.

PROPOSITION: If R is a UFD, then R is normal.

LEMMA: A domain is a UFD if and only if
(1) Every nonzero element has a factorization[1] into irreducibles, and
(2) Every irreducible element generates a prime ideal.

THEOREM: If R is a UFD, then the polynomial ring $R[X]$ is a UFD.

(1) Use the results above to explain why $K[X_1, \dots, X_n]$ (with K a field) and $\mathbb{Z}[X_1, \dots, X_n]$ are normal.

Because fields and $\mathbb{Z}$ are UFDs, so $K[X_1, \dots, X_n]$ and $\mathbb{Z}[X_1, \dots, X_n]$ are UFDs, hence normal.

(2) Prove the Proposition above.

Let $k = a/b$ be in the fraction field of R written in lowest terms. Suppose that k is integral over R and take an equation $k^n + r_1 k^{n-1} + \cdots + r_n = 0$. Plugging in and clearing denominators gives $a^n + r_1 a^{n-1} b + \cdots + r_n b^n = 0$. Then a^n is a multiple of b, so any irreducible factor of b is an irreducible factor of a by unique factorization. The only possibility is that b admits no irreducible factors; i.e., b is a unit, so $k \in R$.

(3) Let K be a module-finite field extension of $\mathbb{Q}$. The **ring of integers** in K, sometimes denoted $\mathcal{O}_K$, is the integral closure of $\mathbb{Z}$ in K.
(a) What is the ring of integers in $\mathbb{Q}(\sqrt{2})$?
(b) For $L = \mathbb{Q}(\sqrt{-3})$, show that $\frac{1+\sqrt{-3}}{2} \in \mathcal{O}_L$. In particular, $\mathcal{O}_L \supsetneq \mathbb{Z}[\sqrt{-3}]$.
(c) Explain why $\mathcal{O}_K$ is normal.
(d) Explain why, if $\mathbb{Z} \subseteq \mathcal{O}_K$ is algebra-finite, then $\mathcal{O}_K \cong \mathbb{Z}^n$ as abelian groups for some $n \in \mathbb{N}$.
(e) Do we have a theorem that implies $\mathbb{Z} \subseteq \mathcal{O}_K$ is algebra-finite?

(a) $\mathbb{Z}[\sqrt{2}]$.
(b) If $\omega = \frac{1+\sqrt{-3}}{2}$, note that $\omega^2 = \frac{-1+1\sqrt{-3}}{2} = \omega - 1$, so $\omega^2 - \omega + 1 = 0$.
(c) If $k \in K$ is integral over $\mathcal{O}_K$, then k is integral over $\mathcal{O}_K$ and hence over $\mathbb{Z}$ (by Corollary 2 from last time). Then by definition, $k \in \mathcal{O}_K$.
(d) If $\mathbb{Z} \subseteq \mathcal{O}_K$ is algebra-finite, then since it is integral, it is also module-finite. $\mathcal{O}_K$ is definitely torsion free, since it's contained in a field, so by the structure theorem for fg abelian groups, it is isomorphic to a finite number of copies of $\mathbb{Z}$.
(e) Not yet!

(4) Discuss the proof of the Lemma above.

We show by induction on n, that for any element $r \in R$ that can has an irreducible factorization as a unit times a product on n irreducibles (counting repetitions), that any other irreducible

[1] i.e., for any $r \in R$, there exists a unit u and a finite (possibly empty) list of irreducibles $a_1, \dots, a_n$ such that $r = u a_1 \cdots a_n$.

factorization agrees with the given one up to associates and reordering. If r is a unit, then any factorization only consists of units, since otherwise r is a divisible by prime element, contradicting that it is a unit.

Say that p is an irreducible in the first factorization of r, so $r = ps$ for some s. Then given any irreducible factorization of r, p must divide some irreducible factor since (p) is prime, and by definition, p must be associate to that irreducible. Then we can cancel p from both factorizations and apply the induction hypothesis to s.

(5) Let K be a field, and $R = K[X^2, XY, Y^2] \subseteq K[X, Y]$. Prove[2] that R is *not* a UFD, but R is normal.

This solution is embargoed.

(6) Prove the Theorem above. You might find it useful to recall the following:
GAUSS' LEMMA: Let R be a UFD and let K be the fraction field of R.
   (a) $f \in R[X]$ is irreducible if and only if f is irreducible in $K[X]$ and the coefficients of f have no common factor.
   (b) Let $r \in R$ be irreducible, and $f, g \in R[X]$. If r divides every coefficient of fg, then either r divides every coefficient of f, or r divides every coefficient of g.

(7) Let R be a normal domain, and s be an element of some domain $S \supseteq R$. Let K be the fraction field of R. Show that if s is integral over R, then the minimal polynomial of s has all of its coefficients in R.

[2]Hint: Use $K[X, Y]$ to your advantage.