DEFINITION: Let R be a domain. The **normalzation** of R is the integral closure of R in $\mathrm{Frac}(R)$. We say that R is **normal** if it is equal to its normalization, i.e., if R is integrally closed in its fraction field.

PROPOSITION: If R is a UFD, then R is normal.

LEMMA: A domain is a UFD if and only if

(1) Every nonzero element has a factorization[1] into irreducibles, and
(2) Every irreducible element generates a prime ideal.

THEOREM: If R is a UFD, then the polynomial ring $R[X]$ is a UFD.

(1) Use the results above to explain why $K[X_1, \ldots, X_n]$ (with K a field) and $\mathbb{Z}[X_1, \ldots, X_n]$ are normal.

(2) Prove the Proposition above.

(3) Let K be a module-finite field extension of $\mathbb{Q}$. The **ring of integers** in K, sometimes denoted $\mathcal{O}_K$, is the integral closure of $\mathbb{Z}$ in K.

- **(a)** What is the ring of integers in $\mathbb{Q}(\sqrt{2})$?
- **(b)** For $L = \mathbb{Q}(\sqrt{-3})$, show that $\frac{1+\sqrt{-3}}{2} \in \mathcal{O}_L$. In particular, $\mathcal{O}_L \supsetneqq \mathbb{Z}[\sqrt{-3}]$.
- **(c)** Explain why $\mathcal{O}_K$ is normal.
- **(d)** Explain why, if $\mathbb{Z} \subseteq \mathcal{O}_K$ is algebra-finite, then $\mathcal{O}_K \cong \mathbb{Z}^n$ as abelian groups for some $n \in \mathbb{N}$.
- **(e)** Do we have a theorem that implies $\mathbb{Z} \subseteq \mathcal{O}_K$ is algebra-finite?

(4) Discuss the proof of the Lemma above.

(5) Let K be a field, and $R = K[X^2, XY, Y^2] \subseteq K[X, Y]$. Prove[2] that R is *not* a UFD, but R is normal.

(6) Prove the Theorem above. You might find it useful to recall the following:
GAUSS' LEMMA: Let R be a UFD and let K be the fraction field of R.

- (a) $f \in R[X]$ is irreducible if and only if f is irreducible in $K[X]$ and the coefficients of f have no common factor.
- (b) Let $r \in R$ be irreducible, and $f, g \in R[X]$. If r divides every coefficient of fg, then either r divides every coefficient of f, or r divides every coefficient of g.

(7) Let R be a normal domain, and s be an element of some domain $S \supseteq R$. Let K be the fraction field of R. Show that if s is integral over R, then the minimal polynomial of s has all of its coefficients in R.

[1] i.e., for any $r \in R$, there exists a unit u and a finite (possibly empty) list of irreducibles $a_1, \ldots, a_n$ such that $r = ua_1 \cdots a_n$.
[2] Hint: Use $K[X, Y]$ to your advantage.