DEFINITION: Let $\phi : A \to R$ be a ring homomorphism. We say that ϕ is **integral** or that R is **integral over** A if every element of R is integral over A.

THEOREM: A homomorphism $\phi : A \to R$ is module-finite if and only if it is algebra-finite and integral. In particular, every module-finite extension is integral.

COROLLARY 1: An algebra generated (as an algebra) by integral elements is integral.

COROLLARY 2: If $R \subseteq S$ is integral, and x is integral over S, then x is integral over R.

PROPOSITION: Let $R \subseteq S$ be an integral extension of domains. Then R is a field if and only if S is a field.

DEFINITION: Let A be a ring, and R be an A-algebra. The **integral closure** of A in R is the set of elements in R that are integral over A.

(1) Proof of Theorem:
   (a) Very briefly explain why, to prove that module-finite implies integral in general, it suffices to show the claim for an inclusion $A \subseteq R$.
   (b) Take a module generating set $\{1, r_2, \dots, r_n\}$ for R as an A-module, and write it as a row vector $v = \begin{bmatrix} 1 & r_2 & \cdots & r_n \end{bmatrix}$. Let $x \in R$. Explain why there is a matrix $M \in \mathrm{Mat}_{n\times n}(A)$ such that $vM = xv$.
   (c) Apply a TRICK to obtain a monic polynomial over A that x satisfies.
   (d) Combine the previous parts with results from last time to complete the proof of the Theorem.

(2) Let $R = \mathbb{C}[X, X^{1/2}, X^{1/3}, \dots] \subseteq \overline{\mathbb{C}(X)}$, where $X^{1/n}$ is an nth root of X. Is $\mathbb{C}[X] \subseteq R$ integral[1]? Is it module-finite? Is it algebra-finite?

(3) Proof of Corollary 1: Let R be an A-algebra.
   (a) If $x, y \in R$ are integral over A, explain why $A[x, y] \subseteq R$ is integral over A. Now explain why $x \pm y$ and xy are integral over A.
   (b) Deduce that the integral closure of A in R is a ring, and moreover an A-subalgebra of R.
   (c) Now let S be a set of integral elements. Apply (b) to the ring $R = A[S]$ in place of R. Complete the proof of the Corollary.

(4) Proof of Proposition:
   (a) First, assume that S is a field, and let $r \in R$ be nonzero. Explain why r has an inverse in S.
   (b) Take an integral equation for $r^{-1} \in S$ over R, and solve for r^{-1} in terms of things in R. Deduce that R must also be a field.
   (c) Now, assume that R is a field, and that S is a domain, and let $s \in S$ be nonzero. Explain why $R[s]$ is a finite-dimensional vector space.
   (d) Explain why the multiplication by s map from $R[s]$ to itself is surjective. Deduce that S must also be a field.

(5) Prove Corollary 2.

[1] You might find the Corollary helpful.

(6) Let $A = \mathbb{C}[X, Y]$ be a polynomial ring, and $R = \dfrac{\mathbb{C}[X, Y, U, V]}{(U^2 - UX + 3X^3, V^2 - 7Y)}$. Find an equation of integral dependence for $U + V$ over A.