DEFINITION: Let $\phi : R \to S$ be a ring homomorphism.

- We say that ϕ is **algebra-finite**, or S is **algebra-finite** over R, if S is a finitely generated R-algebra.
- We say that ϕ is **module-finite**, or S is **module-finite** over R, if S is a finitely generated R-module.

One also often encounters the less self-explanatory terms **finite type** for algebra-finite, and **finite** for module-finite, but we will avoid these.

LEMMA: A module-finite map is algebra-finite. The converse is false.

DEFINITION: Let R be an A-algebra. We say that an element $r \in R$ is **integral** over A if r satisfies a monic polynomial with coefficients in A.

PROPOSITION: Let R be an A-algebra. If $r_1, \dots, r_n \in R$ are integral over A, then $A[r_1, \dots, r_n]$ is module-finite over A.

(1) Algebra-finite vs module-finite: Let $\phi : A \to R$ be a ring homomorphism and $r_1, \dots, r_n \in R$.
- **(a)** Agree or disagree: an A-linear combination of $r_1, \dots, r_n$ is a special type of polynomial expression of $r_1, \dots, r_n$ with coefficients in A.
- **(b)** Explain why $R = \sum_{i=1}^{n} Ar_i$ implies $R = A[r_1, \dots, r_n]$. Explain why module-finite implies algebra-finite.
- **(c)** Let $R = A[X]$ be a polynomial ring in one variable over A. Is the inclusion map $A \subseteq A[X]$ algebra-finite? Module-finite?
- **(d)** Give an example of a map that is module-finite (and hence also algebra-finite).
- **(e)** Give an example of a map that is not algebra-finite (and hence also not module-finite).

> **(a)** Agree.
> **(b)** The first part follows from what you just agreed to.
> **(c)** Algebra-finite but not module-finite.
> **(d)** Possibilities include $\mathbb{Z} \subseteq \mathbb{Z}[\sqrt{2}]$, $\mathbb{R} \subseteq \mathbb{C}$
> **(e)** Possibilities include $\mathbb{Z} \subseteq \mathbb{Q}$, $K \subseteq K[X_1, X_2, \dots]$.

(2) Integral elements: Use the definition of integral to determine whether each is integral or not.
- **(a)** An indeterminate X in a polynomial ring $A[X]$, over A.
- **(b)** $\sqrt[3]{2}$, over $\mathbb{Z}$.
- **(c)** $\frac{1}{2}$, over $\mathbb{Z}$.

> **(a)** No: X satisfies no polynomial over A.
> **(b)** Yes: $\sqrt[3]{2}$ is a root of $T^3 - 2$.
> **(c)** No: given $T^n + a_1 T^{n-1} + \cdots + a_n = 0$ with $a_i \in \mathbb{Z}$, plugging in $T = 1/2$ and clearing denominators gives $1 + 2a_1 + \cdots + 2^n a_n = 0$, which is impossible.

(3) Proof of Proposition: Let A be a ring.
- **(a)** Let $f \in A[X]$ be monic, and let $T = A[X]/(f)$. Explain why T is module-finite over A. What is a generating set?
- **(b)** Let $R = A[r]$ be an algebra generated by one element $r \in R$. Suppose that r satisfies a monic polynomial $f \in A[X]$. How is R related to the ring T as in part (a)? Must they be equal?
- **(c)** Show that R as in (b) is module-finite over A. What is a generating set?

(d) Let $S = A[r_1, \ldots, r_t]$ with $r_1, \ldots, r_t \in S$ integral over A. Use (c) and (4b) below to show that $A \to S$ is module-finite.

(a) We showed earlier that T is a free A-module with basis given by powers of $[X]$ of degree less than the top degree of f.
(b) R is a quotient of T, but could be smaller (a proper quotient). For example, take $R = \mathbb{Z}[X]/(X^2, 2X)$.
(c) It is generates by the powers of $[X]$ of degree than the top degree of f.
(d) This follows from (c), 2(b), and induction.

(4) Finiteness conditions and compositions: Let $R \subseteq S \subseteq T$ be rings.
 (a) If $R \subseteq S$ and $S \subseteq T$ are algebra-finite, show[1] that the composition $R \subseteq T$ is algebra-finite.
 (b) If $R \subseteq S$ and $S \subseteq T$ are module-finite, show[2] that the composition $R \subseteq T$ is module-finite.

(a) If $S = R[s_1, \ldots, s_m]$ and $T = S[t_1, \ldots, t_n]$. We claim that $T = R[s_1, \ldots, s_m, t_1, \ldots, t_n]$. Suppose that $T' \subseteq T$ is an R-subalgebra containing $s_1, \ldots, s_m, t_1, \ldots, t_n$. Since $s_1, \ldots, s_m \in T'$, we have $S \subseteq T'$ so T' is a S-subalgebra of T. But since $t_1, \ldots, t_n \in T'$ we then must have $T' = T$.
(b) If $S = \sum_i Ra_i$ and $T = \sum_j Sb_j$, we claim that $T = \sum_{i,j} Ra_ib_j$. Indeed, given $t \in T$, we can write $t = \sum_j s_jb_j$, and for each s_j we can write $s_j = \sum_i r_{i,j}a_i$, so $t = \sum_j(\sum_i r_{i,j}a_i)b_j$ is an R-linear combination of a_ib_j.

(5) Power series rings:
 (a) Let $A \to R$ be algebra-finite. Show that R is a countably-generated A-module.
 (b) Let A be a ring and $R = A[\![X]\!]$ be a power series ring over A. Show[3] that R is not a countably generated A-module. Deduce that R is not algebra-finite over A.

(a) If $R = A[X_1, \ldots, X_n]$, then R is a free A-module on basis given by monomials. This is a countable set, so R is a countably-generated A-module. In the general case of $A \to R$ be algebra-finite, R is a quotient of a polynomial ring in finitely many variable, so R is a countably-generated A-module.
(b) Suppose $R = \sum_{i=1}^{\infty} Af_i$ is countably generated. Write $[g]_{\leq j}$ for the sum of terms in g of degree at most j and similar things.
We claim that there is some $g \in R$ such that $[g]_{\leq n^2} \notin \sum_{i=1}^{n} A[f_i]_{\leq n^2}$. We construct such g recursively. Suppose we have such a g that satisfies the condition some n. We need to show that there are coefficients $a_{n^2+1}, \ldots, a_{(n+1)^2}$ such that $[g]_{\leq (n+1)^2} \notin \sum_{i=1}^{n+1} A[f_i]_{\leq (n+1)^2}$; we will choose these coefficients with the stronger property that $[g]_{>n\&\leq(n+1)^2} \notin \sum_{i=1}^{n+1} A[f_i]_{>n\&\leq(n+1)^2}$. To do this, just note that $\sum_{i=1}^{n+1} A[f_i]_{>n\&\leq(n+1)^2}$ is a submodule of A^{2n+1} with $n+1$ generators, so is a proper submodule; choose any element of the complement. Thus there exists a g as claimed.

[1] Hint: If $S = R[s_1, \ldots, s_m]$ and $T = S[t_1, \ldots, t_n]$, apply the definition of "algebra generated by" to $R[s_1, \ldots, s_m, t_1, \ldots, t_n] \subseteq T$. Why must the LHS contain S? After that, why must it contain T?
[2] Hint: If $S = \sum_i Rs_i$ and $T = \sum_j St_j$, use the "linear combinations" characterization of module generators to show $T = \sum_{i,j} Rs_it_j$.
[3] Hint: Write $[g]_{\leq j}$ for the sum of terms in g of degree at most j. Suppose $R = \sum_{i=1}^{\infty} Af_i$, and construct $g \in R$ such that $[g]_{\leq n^2} \notin \sum_{i=1}^{n} A[f_i]_{\leq n^2}$.

But then $g \notin \sum_{i=1}^{\infty} Af_i$, since if it were, g would be an A-linear combination of finitely many such f_i, so $g \in \sum_{i=1}^{N} Af_i$ for some N, and hence $[g]_{\leq N^2} \in \sum_{i=1}^{N} A[f_i]_{\leq N^2}$, a contradiction.

It follows from (1) that R is not a finitely-generated A-algebra.

(6) Let $R \subseteq S \subseteq T$ be rings.
  (a) If $R \subseteq T$ is algebra-finite, must $S \subseteq T$ be? What about $R \subseteq S$?
  (b) If $R \subseteq T$ is module-finite, must $S \subseteq T$ be? What[4] about $R \subseteq S$?

(a) $S \subseteq T$ must be, as following immediately from the definition. $R \subseteq S$ need not, e.g., for $K[X] \subseteq K[X, XY, XY^2, \cdots] \subseteq K[X, Y]$.
(b) $S \subseteq T$ must be, as following immediately from the definition. $R \subseteq S$ need not, e.g., for $K[X_1, X_2, \ldots] \subseteq K[X_1, X_2, \ldots] \ltimes (X_1, X_2, \ldots) \subseteq K[X_1, X_2, \ldots] \ltimes K[X_1, X_2, \ldots]$.

(7) Let R be a ring, and M be an R-module. The **Nagata idealization** of M in R, denoted $R \ltimes M$, is the ring that
- as a set and an additive group is just $R \times M = \{(r, m) \mid r \in R, m \in M\}$, and
- has multiplication $(r, m)(s, n) = (rs, rn + sm)$.

Convince yourself that $R \ltimes M$ is an R-algebra. Show that $R \subseteq R \ltimes M$ is module-finite if and only if M is a finitely generated R-module.

[4]Hint: Use a problem below.