DEFINITION: Let $\phi : R \to S$ be a ring homomorphism.

- We say that ϕ is **algebra-finite**, or S is **algebra-finite** over R, if S is a finitely generated R-algebra.
- We say that ϕ is **module-finite**, or S is **module-finite** over R, if S is a finitely generated R-module.

One also often encounters the less self-explanatory terms **finite type** for algebra-finite, and **finite** for module-finite, but we will avoid these.

LEMMA: A module-finite map is algebra-finite. The converse is false.

DEFINITION: Let R be an A-algebra. We say that an element $r \in R$ is **integral** over A if r satisfies a monic polynomial with coefficients in A.

PROPOSITION: Let R be an A-algebra. If $r_1, \dots, r_n \in R$ are integral over A, then $A[r_1, \dots, r_n]$ is module-finite over A.

(1) Algebra-finite vs module-finite: Let $\phi : A \to R$ be a ring homomorphism and $r_1, \dots, r_n \in R$.
- **(a)** Agree or disagree: an A-linear combination of $r_1, \dots, r_n$ is a special type of polynomial expression of $r_1, \dots, r_n$ with coefficients in A.
- **(b)** Explain why $R = \sum_{i=1}^{n} Ar_i$ implies $R = A[r_1, \dots, r_n]$. Explain why module-finite implies algebra-finite.
- **(c)** Let $R = A[X]$ be a polynomial ring in one variable over A. Is the inclusion map $A \subseteq A[X]$ algebra-finite? Module-finite?
- **(d)** Give an example of a map that is module-finite (and hence also algebra-finite).
- **(e)** Give an example of a map that is not algebra-finite (and hence also not module-finite).

(2) Integral elements: Use the definition of integral to determine whether each is integral or not.
- **(a)** An indeterminate X in a polynomial ring $A[X]$, over A.
- **(b)** $\sqrt[3]{2}$, over $\mathbb{Z}$.
- **(c)** $\frac{1}{2}$, over $\mathbb{Z}$.

(3) Proof of Proposition: Let A be a ring.
- **(a)** Let $f \in A[X]$ be monic, and let $T = A[X]/(f)$. Explain why T is module-finite over A. What is a generating set?
- **(b)** Let $R = A[r]$ be an algebra generated by one element $r \in R$. Suppose that r satisfies a monic polynomial $f \in A[X]$. How is R related to the ring T as in part (a)? Must they be equal?
- **(c)** Show that R as in (b) is module-finite over A. What is a generating set?
- **(d)** Let $S = A[r_1, \dots, r_t]$ with $r_1, \dots, r_t \in S$ integral over A. Use (c) and (4b) below to show that $A \to S$ is module-finite.

(4) Finiteness conditions and compositions: Let $R \subseteq S \subseteq T$ be rings.
- (a) If $R \subseteq S$ and $S \subseteq T$ are algebra-finite, show[1] that the composition $R \subseteq T$ is algebra-finite.
- (b) If $R \subseteq S$ and $S \subseteq T$ are module-finite, show[2] that the composition $R \subseteq T$ is module-finite.

[1]Hint: If $S = R[s_1, \dots, s_m]$ and $T = S[t_1, \dots, t_n]$, apply the definition of "algebra generated by" to $R[s_1, \dots, s_m, t_1, \dots, t_n] \subseteq T$. Why must the LHS contain S? After that, why must it contain T?

[2]Hint: If $S = \sum_i Rs_i$ and $T = \sum_j St_j$, use the "linear combinations" characterization of module generators to show $T = \sum_{i,j} Rs_it_j$.

(5) Power series rings:
   (a) Let $A \to R$ be algebra-finite. Show that R is a countably-generated A-module.
   (b) Let A be a ring and $R = A[\![X]\!]$ be a power series ring over A. Show[3] that R is not a countably generated A-module. Deduce that R is not algebra-finite over A.

(6) Let $R \subseteq S \subseteq T$ be rings.
   (a) If $R \subseteq T$ is algebra-finite, must $S \subseteq T$ be? What about $R \subseteq S$?
   (b) If $R \subseteq T$ is module-finite, must $S \subseteq T$ be? What[4] about $R \subseteq S$?

(7) Let R be a ring, and M be an R-module. The **Nagata idealization** of M in R, denoted $R \ltimes M$, is the ring that
   - as a set and an additive group is just $R \times M = \{(r, m) \mid r \in R, m \in M\}$, and
   - has multiplication $(r, m)(s, n) = (rs, rn + sm)$.

   Convince yourself that $R \ltimes M$ is an R-algebra. Show that $R \subseteq R \ltimes M$ is module-finite if and only if M is a finitely generated R-module.

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[3]Hint: Write $[g]_{\leq j}$ for the sum of terms in g of degree at most j. Suppose $R = \sum_{i=1}^{\infty} A f_i$, and construct $g \in R$ such that $[g]_{\leq n^2} \notin \sum_{i=1}^{n} A[f_i]_{\leq n^2}$.

[4]Hint: Use a problem below.