DEFINITION: A module is **Noetherian** if every ascending chain of submodules $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ eventually stabilizes: i.e., there is some N such that $M_n = M_N$ for all $n \geq N$.

THEOREM: If R is a Noetherian ring, then an R-module M is Noetherian if and only M is finitely generated.

COROLLARY: If R is a Noetherian ring, then a submodule of a finitely generated R-module is finitely generated.

LEMMA: Let M be an R-module and $N \subseteq M$ a submodule. Let L, L' be two more submodules of M. Then $L = L'$ if and only if $L \cap N = L' \cap N$ and $\frac{L+N}{N} = \frac{L'+N}{N}$.

(1) Equivalences for Noetherianity.
- **(a)** Explain why M is Noetherian if and only if every submodule of M is finitely generated.
- **(b)** Explain why M is Noetherian if and only if every nonempty collection of submodules has a maximal element.

> **(a)** Analogous to what we did with ideals.
> **(b)** Analogous to what we did with ideals.

(2) Submodules and quotient modules: Let $N \subseteq M$.
- **(a)** Show that if M is a Noetherian R-module, then N is a Noetherian R-module.
- **(b)** Show that if M is a Noetherian R-module, then M/N is a Noetherian R-module.
- **(c)** Use the Lemma above to show that if N and M/N are Noetherian R-modules, then M is a Noetherian R-module.

> **(a)** A chain of submodules of N is a chain of submodules of M, so by hypothesis must stabilize.
> **(b)** The submodules of M/N are in containment-preserving bijection with the submodules of M that contain N, so a chain of submodules of M/N must stabilize.
> **(c)** Suppose we have a chain of submodules M_i of M. By intersecting with N, we get a chain of submodules of $M_i \cap N$ of N, which by hypothesis, must stabilize at some $n = a$. By taking images in M/N, we get a chain of submodules $\frac{M_i+N}{N}$ of M/N that must stabilize at some $n = b$. Then for $n \geq \max\{a, b\}$ by the Lemma, we must have that the chain M_i stabilizes.

(3) Proof of Theorem: Let R be a Noetherian ring.
- **(a)** Explain why R is a Noetherian R-module.
- **(b)** Show that R^n is a Noetherian R-module for every n.
- **(c)** Deduce the Theorem above.
- **(d)** Deduce the Corollary above.

> **(a)** The submodules of R are just the ideals of R.
> **(b)** There is a copy of R^{n-1} in R^n (where the last coordinate is zero) with quotient R^1, so it follows by induction on n.
> **(c)** If M is Noetherian, then every submodule of M including M itself is finitely generated. Conversely, if M is finitely generated, then M is a quotient of R^n for some n, so it follows from (3b) and (2b).
> **(d)** Follows from (3c) and (2a).

(4) Proof of Hilbert Basis Theorem for $R[X]$: Let R be a Noetherian ring.

(a) Let I be an ideal of $R[X]$. Given a nonzero element $f \in R[X]$, set $\mathrm{LT}(f)$ to be the leading coefficient[1] of f and $\mathrm{LT}(0) = 0$, and let $\mathrm{LT}(I) = \{\mathrm{LT}(f) \mid f \in I\}$. Is $\mathrm{LT}(I)$ an ideal of R?

(b) Let $f_1, \dots, f_n \in R[X]$ be such that $\mathrm{LT}(f_1), \dots, \mathrm{LT}(f_n)$ generate $\mathrm{LT}(I)$. Let N be the maximum of the top degrees of f_i. Show that every element of I can be written as $\sum_i r_i f_i + g$ with $r_i, g \in R[X]$ and the top degree of $g \in I$ is less than N.

(c) Write $R[X]_{<N}$ for the R-submodule of $R[X]$ consisting of polynomials with top degree $< N$. Show that $I \cap R[X]_{<N}$ is a finitely generated R-module.

(d) Complete the proof of the Theorem.

(a) Yes; we just check the definition.

(b) We proceed by induction on top degree of $f \in I$. For f with top degree less than N, we just take $g = f$ and $r_i = 0$. For f with top degree $t \geq N$, write $f = aX^t +$ lower degree terms, and $a = \sum_i a_i \mathrm{LT}(f_i)$. Then $\sum_i a_i X^{t-n_i} f_i = aX^t +$ lower degree terms, so $f' = f - \sum_i a_i X^{t-n_i} f_i \in I$ is of lower degree. We can then write f' in the desired form by induction, and then the original f as well.

(c) $I \cap R[X]_{<N}$ is an R-submodule of $R[X]_{<N}$, which is generated by $1, X, \dots, X^{N-1}$, whence finitely generated. Since R is Noetherian, this submodule is also Noetherian.

(d) Fix an R-module generating set $g_1, \dots, g_s$ for $I \cap R[X]_{<N}$. We claim that $I = (f_1, \dots, f_n, g_1, \dots, g_s)$. By construction we have $\supseteq$. Then, given $f \in I$, we can write $f = \sum_i r_i f_i + g$ and $g = \sum_j a_j g_j$ with $a_j \in R$, so $f \in (f_1, \dots, f_n, g_1, \dots, g_s)$. Thus, I is finitely generated.

(5) Proof of Hilbert Basis Theorem for $R[\![X]\!]$: How can you modify the Proof of Hilbert Basis Theorem for $R[X]$ to work in the power series case? Make it happen!

We use lowest degree terms instead. Define $\mathrm{LT}(f)$ to be the bottom coefficient of f. Proceeding similarly, we can show that if $f_1, \dots, f_n \in R[\![X]\!]$ are such that $\mathrm{LT}(f_1), \dots, \mathrm{LT}(f_n)$ generate $\mathrm{LT}(I)$, then and $f \in I$ can be written as $\sum_i r_i f_i + g$ with g a *polynomial* in X of top degree less than N, and continue as in the polynomial case.

(6) Prove the Lemma.

(7) Noetherianity and module-finite inclusions: Let $R \subseteq S$ be module-finite.

(a) Without using the Hilbert Basis Theorem, show that is R is Noetherian, then S is Noetherian.

(b) EAKIN-NAGATA THEOREM: Show that if S is Noetherian, then R is Noetherian.

[1] That is, if $f = \sum_i a_i X^i$ and $k = \max\{i \mid a_i \neq 0\}$, then $\mathrm{LT}(f) = a_k$.