DEFINITION: A module is **Noetherian** if every ascending chain of submodules $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ eventually stabilizes: i.e., there is some N such that $M_n = M_N$ for all $n \geq N$.

THEOREM: If R is a Noetherian ring, then an R-module M is Noetherian if and only M is finitely generated.

COROLLARY: If R is a Noetherian ring, then a submodule of a finitely generated R-module is finitely generated.

LEMMA: Let M be an R-module and $N \subseteq M$ a submodule. Let L, L' be two more submodules of M. Then $L = L'$ if and only if $L \cap N = L' \cap N$ and $\frac{L+N}{N} = \frac{L'+N}{N}$.

(1) Equivalences for Noetherianity.
- **(a)** Explain why M is Noetherian if and only if every submodule of M is finitely generated.
- **(b)** Explain why M is Noetherian if and only if every nonempty collection of submodules has a maximal element.

(2) Submodules and quotient modules: Let $N \subseteq M$.
- **(a)** Show that if M is a Noetherian R-module, then N is a Noetherian R-module.
- **(b)** Show that if M is a Noetherian R-module, then M/N is a Noetherian R-module.
- **(c)** Use the Lemma above to show that if N and M/N are Noetherian R-modules, then M is a Noetherian R-module.

(3) Proof of Theorem: Let R be a Noetherian ring.
- **(a)** Explain why R is a Noetherian R-module.
- **(b)** Show that R^n is a Noetherian R-module for every n.
- **(c)** Deduce the Theorem above.
- **(d)** Deduce the Corollary above.

(4) Proof of Hilbert Basis Theorem for $R[X]$: Let R be a Noetherian ring.
- **(a)** Let I be an ideal of $R[X]$. Given a nonzero element $f \in R[X]$, set $\mathrm{LT}(f)$ to be the leading coefficient[1] of f and $\mathrm{LT}(0) = 0$, and let $\mathrm{LT}(I) = \{\mathrm{LT}(f) \mid f \in I\}$. Is $\mathrm{LT}(I)$ an ideal of R?
- **(b)** Let $f_1, \dots, f_n \in R[X]$ be such that $\mathrm{LT}(f_1), \dots, \mathrm{LT}(f_n)$ generate $\mathrm{LT}(I)$. Let N be the maximum of the top degrees of f_i. Show that every element of I can be written as $\sum_i r_i f_i + g$ with $r_i, g \in R[X]$ and the top degree of $g \in I$ is less than N.
- **(c)** Write $R[X]_{<N}$ for the R-submodule of $R[X]$ consisting of polynomials with top degree $< N$. Show that $I \cap R[X]_{<N}$ is a finitely generated R-module.
- **(d)** Complete the proof of the Theorem.

(5) Proof of Hilbert Basis Theorem for $R[\![X]\!]$: How can you modify the Proof of Hilbert Basis Theorem for $R[X]$ to work in the power series case? Make it happen!

(6) Prove the Lemma.

(7) Noetherianity and module-finite inclusions: Let $R \subseteq S$ be module-finite.
- (a) Without using the Hilbert Basis Theorem, show that is R is Noetherian, then S is Noetherian.
- *(b)* EAKIN-NAGATA THEOREM: Show that if S is Noetherian, then R is Noetherian.

[1]That is, if $f = \sum_i a_i X^i$ and $k = \max\{i \mid a_i \neq 0\}$, then $\mathrm{LT}(f) = a_k$.