Recall that given matrices A and B, the matrix product AB consists of linear combinations, namely: Each column of AB is a linear combinations of the columns of A, with coefficients/weights coming from the corresponding columns of B. That is,

$$\big(\text{col } j \text{ of } AB\big) = \sum_{i=1}^{t} b_{ij} \cdot \big(\text{col } i \text{ of } A\big);$$

note that $b_{1j}, \dots, b_{tj}$ is the j-th column of B.

PROPERTIES OF $\det$: For a ring R, the determinant is a function $\det : \mathrm{Mat}_{n\times n}(R) \to R$ such that:

(1) $\det$ is a polynomial expression of the entries of A of degree n.
(2) $\det$ is a linear function of each column.
(3) $\det(A) = 0$ if the columns are linearly dependent.
(4) $\det(AB) = \det(A)\det(B)$.
(5) $\det$ can be computed by Laplace expansion along a row/column.
(6) $\det(A) = \det(A^{\mathrm{tr}})$.
(7) If $\phi : R \to S$ is a ring homomorphism, and $\phi(A)$ is the matrix obtained from A by applying ϕ to each entry, then $\det(\phi(A)) = \phi(\det(A))$.

ADJOINT TRICK: For an $n \times n$ matrix A over R,

$$\det(A)\mathbb{1}_n = A^{\mathrm{adj}} A = A\, A^{\mathrm{adj}},$$

where $(A^{\mathrm{adj}})_{ij} = (-1)^{i+j} \det(\text{matrix obtained from } A \text{ by removing row } j \text{ and column } i)$.

EIGENVECTOR TRICK: Let A be an $n \times n$ matrix, $v \in R^n$, and $r \in R$. If $Av = rv$, then $\det(r\mathbb{1}_n - A)v = 0$. Likewise, if instead v is a row vector and $vA = rv$, then $\det(r\mathbb{1}_n - A)v = 0$.

DEFINITION: Given an $n \times m$ matrix A and $1 \le t \le \min\{m, n\}$ the **ideal of** $t \times t$ **minors of** A, denoted $I_t(A)$, is the ideal generated by the determinants of all $t \times t$ submatrices of A given by choosing t rows and t columns. For $t = 0$, we set $I_0(A) = R$ and for $t > \min\{m, n\}$ we set $I_t(A) = 0$.

LEMMA: If A is an $n \times m$ matrix, B is an $m \times \ell$ matrix, and $t \le 1$, then

- $I_{t+1}(A) \subseteq I_t(A)$
- $I_t(AB) \subseteq I_t(A) \cap I_t(B)$.

PROPOSITION: Let M be a finitely presented module. Suppose that A is an $n \times m$ presentation matrix for M. Then $I_n(A)M = 0$. Conversely, if $fM = 0$, then $f^n \in I_n(A)$.

(1) Let M be a module. Suppose that $m_1, \dots, m_n$ is a generating set with corresponding presentation matrix A. Which of the following is true:

$$A \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} \stackrel{?}{=} 0 \qquad\qquad \begin{bmatrix} m_1 & \cdots & m_n \end{bmatrix} A \stackrel{?}{=} 0.$$

Explain your answer in terms of the recollection on matrix multiplication above.

The second one!

(2) Eigenvector Trick:

(a) What familiar fact/facts from linear algebra (over fields) is/are related to the Eigenvector Trick?

(b) Use the Adjoint Trick to prove the Eigenvector Trick.

(a) Over a field, an eigenvalue of a matrix is a root of the characteristic polynomial.

(b) If $Av = rv$, then $(A - r\mathbb{1}_n)v = 0$, so multiply by $(A - r\mathbb{1}_n)^{\mathrm{adj}}$ to get $\det(A - r\mathbb{1}_n)v = (A - r\mathbb{1}_n)^{\mathrm{adj}}(A - r\mathbb{1}_n)v = 0$. Likewise on the other side.

(3) Show that a square matrix over a ring R is invertible if and only if its determinant is a unit.

If $AB = \mathbb{1}_n$, then $\det(A)\det(B) = \det(\mathbb{1}_n) = 1$, so $\det(A)$ is a unit. On the other hand, if $\det(A)$ is a unit, then $B = \det(A)^{-1}A^{\mathrm{adj}}$ is an inverse of A by the adjoint trick.

(4) Proof of Proposition:

(a) First consider the case $m = n$. Show that $\det(A)$ kills each generator m_i, and conclude that $I_n(A)M = 0$.

(b) Now consider the case $n \leq m$. Show that for any $n \times n$ submatrix A' of A that $\det(A')M = 0$, and conclude that $I_n(A)M = 0$. What's the deal when $m < n$?

(c) For the "conversely" statement, show that if $fM = 0$ then there is some matrix B such that $AB = f\mathbb{1}_n$, and deduce that $f \in I_n(A)^n$.

(a) Since A is a presentation matrix for M, with the corresponding generating set $m_1, \dots, m_n$, we have $\begin{bmatrix} m_1 & \dots & m_n \end{bmatrix} A = 0$. By the adjoint trick, $\det(A)\begin{bmatrix} m_1 & \dots & m_n \end{bmatrix} = 0$, so $\det(A)$ kills each generator of M. Thus, $\det(A)$ kills M. By definition $I_n(A) = (\det(A))$, so we are done.

(b) Suppose $n \leq m$ and fix m columns of A to form an $n \times n$ submatrix A'. The columns of A' are still relations on $m_1, \dots, m_n$, so the same argument shows that $\det(A')$ kills M. Now, by definition, $I_n(A)$ is generated by the determinants of the submatrices A', so $I_n(A)M = 0$.
When $m < n$, $I_n(A) = 0$, which very much kills M.

(c) If $fM = 0$, then the vector with f in the ith entry and zeroes elsewhere is a relation on the generators, so by definition of presentation matrix, this vector is a linear combination of the columns of A. Thus each column $f\mathbb{1}_n$ is a linear combination of the columns of A, which means that we can write $f\mathbb{1}_n = AB$ for some matrix B following the discussion above. By the Lemma, we have $f^n = \det(f\mathbb{1}_n) \in I_n(AB) \subseteq I_n(A)$. This completes the proof.

(5) Prove the Lemma above.

The first statement follows from Laplace expansion. For the second, it suffices to show that the determinant of any $t \times t$ submatrix of AB is a linear combination of determinants of $t \times t$ submatrices of A; the claim for B follows by applying transposes. We can restrict to the relevant rows of A and columns of B, so we can assume that A is $t \times n$ and B is $n \times t$ for some $n \geq t$. Then AB is a matrix whose columns are linear combinations of the columns of A. Then using linearity of $\det$ in each column, we can write $\det(AB)$ as a linear combination of the determinants of matrices with columns from A, which shown the claim.

(6) Prove[1] FITTING'S LEMMA: If A and B are presentation matrices for the same R-module M of size $n \times m$ and $n' \times m'$ (respectively), and $t \geq 0$, then $I_{n-t}(A) = I_{n'-t}(B)$.

[1] Hint: First consider the case when the two presentations have the same generating sets, but different generating sets for the relations. Reduce to the case where $B = [A|v]$ for a single column v.