EXAMPLE: For a ring R, the following are sources of modules:

(1) The free module of n-tuples R^n, or more generally, for a set Λ, the free module
$$R^{\oplus\Lambda} = \{(r_\lambda)_{\lambda\in\Lambda} \mid r_\lambda \neq 0 \text{ for at most finitely many } \lambda \in \Lambda\}.$$
(2) Every ideal $I \subseteq R$ is a submodule of R.
(3) Every quotient ring R/I is a quotient module of R.
(4) If S is an R-algebra, (i.e., there is a ring homomorphism $\alpha : R \to S$), then S is an R-module by **restriction of scalars**: $r \cdot s := \alpha(r)s$.
(5) More generally, if S is an R-algebra and M is an S-module, then M is also an R-module by **restriction of scalars**: $r \cdot m := \alpha(r) \cdot m$.
(6) Given an R-module M and $m_1, \dots, m_n \in M$, the **module of R-linear relations** on $m_1, \dots, m_n$ is the set of n-tuples $[r_1, \dots, r_n]^{\text{tr}} \in R^n$ such that $\sum_i r_i m_i = 0$ in R.

DEFINITION: Let M be an R-module. Let S be a subset of M. The **submodule generated by** S, denoted[1] $\sum_{m\in S} Rm$, is the smallest R-submodule of M containing S. Equivalently,
$$\sum_{m\in S} Rm = \left\{ \sum r_i m_i \mid r_i \in R, m_i \in S \right\} \quad \text{is the set of } R\text{-linear combinations of elements of } S.$$
We say that S **generates** M if $M = \sum_{m\in S} Rm$.

DEFINITION: A[2] **presentation** of an R-module M consists of a set of generators $m_1, \dots, m_n$ of M as an R-module and a set of generators $v_1, \dots, v_m \in R^n$ for the submodule of R-linear relations on $m_1, \dots, m_n$. We call the $n \times m$ matrix with columns $v_1, \dots, v_m$ a **presentation matrix** for M.

LEMMA: If M is an R-module, and A an $n \times m$ presentation matrix[3] for M, then $M \cong R^n/\text{im}(A)$. We call the module $R^n/\text{im}(A)$ the **cokernel** of the matrix A.

(1) Let M be an R-module and $m_1, \dots, m_n \in M$.
- **(a)** Briefly explain why the characterizations of the submodule generated by S are equivalent.
- **(b)** Briefly explain why $\sum_i Rm_i$ is the image of the R-module homomorphism $\beta : R^n \to M$ such[4] that $\beta(e_i) = m_i$.
- **(c)** Let I be an ideal of R. How does a generating set of I as an ideal compare to a generating set of I as an R-module?
- **(d)** Explain why the Lemma above is true.
- **(e)** If M has an $a \times b$ presentation matrix A, how many generators and how many (generating) relations are in the presentation corresponding to A?
- **(f)** What is a presentation matrix for a free module?

(a) $(\subseteq)$: The elements of the form $\sum r_i m_i$ form a submodule of M that contains S. $(\supseteq)$: A submodule that contains S must also contain the elements of the form $\sum r_i m_i$.

[1] If $S = \{m\}$ is a singleton, we just write Rm, and if $S = \{m_1, \dots, m_n\}$, we may write $\sum_i Rm_i$.
[2] As written, there is a finite set of generators, and a finite set of generators for their relations. This is called a **finite presentation**. One could do the same thing with an infinite generating set and/or infinite generating set for the relations.
[3] $\text{im}(A)$ denotes the **image** or column space of A in R^n. This is equal to the module generated by the columns of A.
[4] where e_i is the vector with ith entry one and all other entries zero.

(b) This is just unpackaging $\mathrm{im}(\beta)$: $\beta((r_1,\ldots,r_n)) = \beta(\sum_i r_i e_i) = \sum_i r_i m_i$.
(c) They are the same.
(d) Follows from (b) and First Isomorphism Theorem.
(e) There are a generators and b relations.
(f) A matrix is free if and only if it has zero presentation matrix.

(2) Describe $\mathbb{Z}[\sqrt{2}]$ as a $\mathbb{Z}$-module.

$Z[\sqrt{2}]$ is a free $\mathbb{Z}$-module with basis $1, \sqrt{2}$.

(3) Module structure for polynomial rings and quotients:
(a) Let $R = A[X]$ be a polynomial ring. Give a generating set for R as an A-module. Is R a free A-module?
(b) Let $R = A[X,Y]$ be a polynomial ring. Give a generating set for R as an A-module. Is R a free A-module?
(c) Let $R = A[X]/(f)$, where f is a monic polynomial of top degree d. Apply the Division Algorithm to show that R is a free A-module with basis $[1], [X], \ldots, [X^{d-1}]$.
(d) Let $R = \mathbb{C}[X,Y]/(Y^3 - iXY + 7X^4)$. Describe R as a $\mathbb{C}[X]$-module, and then give a $\mathbb{C}$-vector space basis.

(a) R is free on basis $1, X, X^2, \ldots$.
(b) R is free on basis $1, X, X^2, \ldots, Y, XY, XY^2, \ldots, Y^2, XY^2, X^2Y^2, \ldots \ldots$.
(c) We need to show that any $[g] \in R$ has a unique expression as an A-linear combination of $[1], \ldots, [X^{d-1}]$. Given $[g]$, take a represenatative g; use the division algorithm to write $g = qf + r$ with top deg r ¡ d. Thus $[g] = [r]$, and since $r \in A1 + AX + \cdots + AX^{d-1}$, $[g] = [r] \in A[1] + \cdots + A[X^{d-1}]$. For uniqueness, it suffices to show linear independence of $[1], \ldots, [X^{d-1}]$; a nontrivial relation would yield a multiple of f in $A[X]$ of degree less than d, which cannot happen.
(d) R is free over $\mathbb{C}[X]$ on $[1], [Y], [Y^2]$. It has as a vector space basis $\{[X^iY^j] \mid i \geq 0, j \in \{0,1,2\}.\}$.

(4) Let $R = \mathbb{C}[X]$ and $S = \mathbb{C}[X, X^{-1}] \subseteq \mathbb{C}(X)$. Find a generating set for S as an R-module. Does there exist a finite generating set for S as an R-module? Is S a free R-module?

S is generated by $\{1/X^n \mid n \geq 0\}$. S cannot be generated by a finite set: if $S = Rf_1 + \cdots + Rf_n$, among $f_1, \ldots, f_n$ there is a largest power of X in the denominator, say m. Then $S \subseteq R\frac{1}{X^m}$, but $\frac{1}{X^{m+1}} \in S \setminus R\frac{1}{X^m}$. S is not free: if it were, there would be a basis element s, and $s \notin xS$, as this would lead to a nontrivial relation with other basis elements, but $S = xS$, so this is impossible.

(5) Presentations of modules: Let K be a field, and $R = K[X,Y]$ be a polynomial ring.
(a) Consider the quotient ring $K \cong R/(X,Y)$ as an R-module. Find a presentation for K as an R-module.
(b) Consider the ideal $I = (X,Y)$ as an R-module. Find a presentation for I as an R-module.
(c) Consider the ideal $J = (X^2, XY, Y^2)$ as an R-module. Find a presentation for J as an R-module.

(a) $[1]$ generates K, and X, Y are the defining relations. So, a presentation matrix is $[X, Y]$.
(b) A generating set is $\{X, Y\}$. To find the relations, suppose that $fX + gY = 0$. Then $fX = -gY$. Writing out $f, -g$ in terms of monomials, one sees that $-g$ must be a multiple of X and f must be a multiple of Y so $f = hY, -g = jX$. Then $hXY = jXY$, so $j = h$. Thus, the relation $\begin{bmatrix} f \\ g \end{bmatrix}$ can be written as $h\begin{bmatrix} Y \\ -X \end{bmatrix}$. A defining relation (and hence the presentation matrix) is $\begin{bmatrix} Y \\ -X \end{bmatrix}$.
(c) A generating set is $\{X^2, XY, Y^2\}$. We have relations $\begin{bmatrix} Y \\ -X \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ Y \\ -X \end{bmatrix}$ corresponding to $Y(X^2) - X(XY) = 0$ and $Y(XY) - X(Y^2) = 0$. We claim that these generate. Suppose that $aX^2+bXY+CY^2 = 0$; we want to show that $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \operatorname{im} \begin{bmatrix} Y & 0 \\ -X & Y \\ 0 & -X \end{bmatrix}$. We can write $a = a'Y + a''$ with $a'' \in K[X]$ and subtracting $a' \begin{bmatrix} Y \\ -X \\ 0 \end{bmatrix}$, we obtain a relation with $a \in K[X]$; similarly, we can assume $c \in K[Y]$. Then plugging in $a(X)X^2 + b(X,Y)XY + c(Y)Y^2$, since each sum has no possible monomials in common, we must have $a = b = c = 0$. This shows the claim.

(6) Let M be an R-module, $S \subseteq M$ a generating set, and $r \in R$. Show that $rM = 0$ if and only if $rm = 0$ for all $m \in S$.

The forward direction is clear. For the other, writing $m = \sum_i r_i m_i$ with $m_i \in S$, if $rm_i = 0$, then $rm = 0$.

(7) Let K be a field, $S = K[X,Y]$ be a polynomial ring, and $R = K[X^2, XY, Y^2] \subseteq S$. Find an R-module M such that $S = R \oplus M$ as R-modules. Given a presentations for S and M as R-modules.

We can take M to be the collection of polynomials all of whose terms have odd degree. Note that M is indeed closed under multiplication by R. A presentation matrix for M is $\begin{bmatrix} XY & Y^2 \\ -X^2 & -XY \end{bmatrix}$ and for S is $\begin{bmatrix} 0 & 0 \\ XY & Y^2 \\ -X^2 & -XY \end{bmatrix}$.

(8) Messing with presentation matrices: Let M be a module with an $n \times m$ presentation matrix A.
(a) If you add a column of zeroes to A, how does M change?
(b) If you add a row of zeroes to A, how does M change?
(c) If you add a row and column to A, with a 1 in the corner and zeroes elsewhere in the new row and column, how does M change?
(d) If A is a block matrix $\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$, what does this say about M?

(a) It doesn't.
(b) Corresponds to adding a free copy of R as a direct sum.
(c) It doesn't.
(d) $M \cong \operatorname{coker}(B) \oplus \operatorname{coker}(C)$