EXAMPLE: For a ring R, the following are sources of modules:

(1) The free module of n-tuples R^n, or more generally, for a set Λ, the free module
$$R^{\oplus \Lambda} = \{(r_\lambda)_{\lambda \in \Lambda} \mid r_\lambda \neq 0 \text{ for at most finitely many } \lambda \in \Lambda\}.$$
(2) Every ideal $I \subseteq R$ is a submodule of R.
(3) Every quotient ring R/I is a quotient module of R.
(4) If S is an R-algebra, (i.e., there is a ring homomorphism $\alpha : R \to S$), then S is an R-module by **restriction of scalars**: $r \cdot s := \alpha(r)s$.
(5) More generally, if S is an R-algebra and M is an S-module, then M is also an R-module by **restriction of scalars**: $r \cdot m := \alpha(r) \cdot m$.
(6) Given an R-module M and $m_1, \ldots, m_n \in M$, the **module of R-linear relations** on $m_1, \ldots, m_n$ is the set of n-tuples $[r_1, \ldots, r_n]^{\text{tr}} \in R^n$ such that $\sum_i r_i m_i = 0$ in R.

DEFINITION: Let M be an R-module. Let S be a subset of M. The **submodule generated by** S, denoted[1] $\sum_{m \in S} Rm$, is the smallest R-submodule of M containing S. Equivalently,
$$\sum_{m \in S} Rm = \left\{ \sum r_i m_i \mid r_i \in R, m_i \in S \right\} \quad \text{is the set of } R\text{-linear combinations of elements of } S.$$
We say that S **generates** M if $M = \sum_{m \in S} Rm$.

DEFINITION: A[2] **presentation** of an R-module M consists of a set of generators $m_1, \ldots, m_n$ of M as an R-module and a set of generators $v_1, \ldots, v_m \in R^n$ for the submodule of R-linear relations on $m_1, \ldots, m_n$. We call the $n \times m$ matrix with columns $v_1, \ldots, v_m$ a **presentation matrix** for M.

LEMMA: If M is an R-module, and A an $n \times m$ presentation matrix[3] for M, then $M \cong R^n/\text{im}(A)$. We call the module $R^n/\text{im}(A)$ the **cokernel** of the matrix A.

(1) Let M be an R-module and $m_1, \ldots, m_n \in M$.
- **(a)** Briefly explain why the characterizations of the submodule generated by S are equivalent.
- **(b)** Briefly explain why $\sum_i Rm_i$ is the image of the R-module homomorphism $\beta : R^n \to M$ such[4] that $\beta(e_i) = m_i$.
- **(c)** Let I be an ideal of R. How does a generating set of I as an ideal compare to a generating set of I as an R-module?
- **(d)** Explain why the Lemma above is true.
- **(e)** If M has an $a \times b$ presentation matrix A, how many generators and how many (generating) relations are in the presentation corresponding to A?
- **(f)** What is a presentation matrix for a free module?

(2) Describe $\mathbb{Z}[\sqrt{2}]$ as a $\mathbb{Z}$-module.

[1] If $S = \{m\}$ is a singleton, we just write Rm, and if $S = \{m_1, \ldots, m_n\}$, we may write $\sum_i Rm_i$.
[2] As written, there is a finite set of generators, and a finite set of generators for their relations. This is called a **finite presentation**. One could do the same thing with an infinite generating set and/or infinite generating set for the relations.
[3] $\text{im}(A)$ denotes the **image** or column space of A in R^n. This is equal to the module generated by the columns of A.
[4] where e_i is the vector with ith entry one and all other entries zero.

(3) Module structure for polynomial rings and quotients:

- **(a)** Let $R = A[X]$ be a polynomial ring. Give a generating set for R as an A-module. Is R a free A-module?
- **(b)** Let $R = A[X, Y]$ be a polynomial ring. Give a generating set for R as an A-module. Is R a free A-module?
- **(c)** Let $R = A[X]/(f)$, where f is a monic polynomial of top degree d. Apply the Division Algorithm to show that R is a free A-module with basis $[1], [X], \dots, [X^{d-1}]$.
- **(d)** Let $R = \mathbb{C}[X, Y]/(Y^3 - iXY + 7X^4)$. Describe R as a $\mathbb{C}[X]$-module, and then give a $\mathbb{C}$-vector space basis.

(4) Let $R = \mathbb{C}[X]$ and $S = \mathbb{C}[X, X^{-1}] \subseteq \mathbb{C}(X)$. Find a generating set for S as an R-module. Does there exist a finite generating set for S as an R-module? Is S a free R-module?

(5) Presentations of modules: Let K be a field, and $R = K[X, Y]$ be a polynomial ring.

- (a) Consider the quotient ring $K \cong R/(X, Y)$ as an R-module. Find a presentation for K as an R-module.
- (b) Consider the ideal $I = (X, Y)$ as an R-module. Find a presentation for I as an R-module.
- (c) Consider the ideal $J = (X^2, XY, Y^2)$ as an R-module. Find a presentation for J as an R-module.

(6) Let M be an R-module, $S \subseteq M$ a generating set, and $r \in R$. Show that $rM = 0$ if and only if $rm = 0$ for all $m \in S$.

(7) Let K be a field, $S = K[X, Y]$ be a polynomial ring, and $R = K[X^2, XY, Y^2] \subseteq S$. Find an R-module M such that $S = R \oplus M$ as R-modules. Given a presentations for S and M as R-modules.

(8) Messing with presentation matrices: Let M be a module with an $n \times m$ presentation matrix A.

- (a) If you add a column of zeroes to A, how does M change?
- (b) If you add a row of zeroes to A, how does M change?
- (c) If you add a row and column to A, with a 1 in the corner and zeroes elsewhere in the new row and column, how does M change?
- (d) If A is a block matrix $\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$, what does this say about M?