DEFINITION: Let A be a ring. An **A-algebra** is a ring R equipped with a ring homomorphism $\phi : A \to R$; we call ϕ the **structure morphism** of the algebra[1]. A **homomorphism** of A-algebras is a ring homomorphism that is compatible with the structure morphisms; i.e., if $\phi : A \to R$ and $\psi : A \to S$ are A-algebras, then $\alpha : R \to S$ is an A-algebra homomorphism if $\alpha \circ \phi = \psi$.

UNIVERSAL PROPERTY OF POLYNOMIAL RINGS: Let[2] A be a ring, and $T = A[X_1, \dots, X_n]$ be a polynomial ring. For any A-algebra R, and any collection of elements $r_1, \dots, r_n \in R$, there is a unique A-algebra homomorphism $\alpha : T \to R$ such that $\alpha(X_i) = r_i$.

DEFINITION: Let A be a ring, and R be an A-algebra. Let S be a subset of R. The **subalgebra generated by** S, denoted $A[S]$, is the smallest A-subalgebra of R containing S. Equivalently[3],

$$A[r_1, \dots, r_n] = \left\{ \sum_{\text{finite}} a r_1^{d_1} \cdots r_n^{d_n} \mid a \in \phi(A) \right\}.$$

DEFINITION: Let R be an A-algebra. Let $r_1, \dots, r_n \in R$. The ideal of **A-algebraic relations** on $r_1, \dots, r_n$ is the set of polynomials $f(X_1, \dots, X_n) \in A[X_1, \dots, X_n]$ such that $f(r_1, \dots, r_n) = 0$ in R. Equivalently, the ideal of A-algebraic relations on $r_1, \dots, r_n$ is the kernel of the homomorphism $\alpha : A[X_1, \dots, X_n] \to R$ given by $\alpha(X_i) = r_i$. We say that a set of elements in an A-algebra is **algebraically independent over** A if it has no nonzero A-algebraic relations.

DEFINITION: A **presentation** of an A-algebra R consists of a set of generators $r_1, \dots, r_n$ of R as an A-algebra and a set of generators $f_1, \dots, f_m \in A[X_1, \dots, X_n]$ for the ideal of A-algebraic relations on $r_1, \dots, r_n$. We call $f_1, \dots, f_m$ a set of **defining relations** for R as an A-algebra.

PROPOSITION: If R is an A-algebra, and $f_1, \dots, f_m$ is a set of defining relations for R as an A-algebra, then $R \cong A[X_1, \dots, X_n]/(f_1, \dots, f_m)$.

(1) Let R be an A-algebra and $r_1, \dots, r_n \in R$.
- **(a)** Discuss why the equivalent characterizations in the definition of $A[r_1, \dots, r_n]$ are equivalent.
- **(b)** Explain why $A[r_1, \dots, r_n]$ is the image of the A-algebra homomorphism $\alpha : A[X_1, \dots, X_n] \to R$ such that $\alpha(X_i) = r_i$.
- **(c)** Suppose that $R = A[r_1, \dots, r_n]$ and let $f_1, \dots, f_m$ be a set of generators for the kernel of the map α. Explain why $R \cong A[X_1, \dots, X_n]/(f_1, \dots, f_m)$, i.e., why the Proposition above is true.
- **(d)** Suppose that R is generated as an A-algebra by a set S. Let I be an ideal of R. Explain why R/I is generated as an A-algebra by the image of S in R/I.
- **(e)** Let $R = A[X_1, \dots, X_n]/(f_1, \dots, f_m)$, where $A[X_1, \dots, X_n]$ is a polynomial ring over A. Find a presentation for R.

[1] Note: the same R with different ϕ's yield different A-algebras. Despite this we often say "Let R be an A-algebra" without naming the structure morphism.

[2] This is equally valid for polynomial rings in infinitely many variables $T = A[X_\lambda \mid \lambda \in \Lambda]$ with a tuple of elements of $\{r_\lambda\}_{\lambda \in \Lambda}$ in R in bijection with the variable set. I just wrote this with finitely many variables to keep the notation for getting too overwhelming.

[3] Again written with a finite set just for convenience.

(2) Presentations of some subrings:

- **(a)** Consider the $\mathbb{Z}$-subalgebra of $\mathbb{C}$ generated by $\sqrt{2}$. Write the notation for this ring. Is there a more compact description of the set of elements in this ring? Find a presentation.
- **(b)** Same as (a) with $\sqrt[3]{2}$ instead of $\sqrt{2}$.
- **(c)** Let K be a field, and $T = K[X, Y]$. Come up with a concrete description of the ring $R = K[X^2, XY, Y^2] \subseteq T$, (i.e., describe in simple terms which polynomials are elements of R), and give a presentation as a K-algebra.

(3) Infinitely generated algebras:

- **(a)** Show that $\mathbb{Q} = \mathbb{Z}[1/p \mid p \text{ is a prime number}]$.
- **(b)** True or false: It is a direct consequence of the conclusion of (a) and the fact that there are infinitely many primes that $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-algebra.
- **(c)** Given $p_1, \dots, p_m$ prime numbers, describe the elements of $\mathbb{Z}[1/p_1, \dots, 1/p_m]$ in terms of their prime factorizations. Can you ever have $\mathbb{Z}[1/p_1, \dots, 1/p_m] = \mathbb{Q}$ for a finite set of primes?
- **(d)** Show that $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-algebra.
- (e) Show that, for a field K, the algebra $K[X, XY, XY^2, XY^3, \dots] \subseteq K[X, Y]$ is not a finitely generated K-algebra.
- (f) Show that, for a field K, the algebra $K[X, Y/X, Y/X^2, Y/X^3, \dots] \subseteq K(X, Y)$ is not a finitely generated K-algebra.

(4) More algebras:

- (a) Give two different nonisomorphic $\mathbb{C}[X]$-algebra structures on $\mathbb{C}$.
- (b) Find a $\mathbb{C}$-algebra generating set for the ring of polynomials in $\mathbb{C}[X, Y]$ that only have terms whose total degree (X-exponent plus Y-exponent) is a multiple of three (e.g., $X^3 + \pi X^5 Y + 5$ is in while $X^3 + \pi X^4 Y + 5$ is out).
- (c) Find a $\mathbb{C}$-algebra presentation for $\mathbb{C} \times \mathbb{C}$.

(5) Let K be a field. Describe which elements are in the K-algebra $K[X, X^{-1}] \subseteq K(X)$, and find an element of $K(X)$ not in $K[X, X^{-1}]$. Then compute[4] a presentation for $K[X, X^{-1}]$ as a K-algebra.

(6) Can you guess defining relations for the ring in (4b)? Can you prove your guess?

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[4] Hint: Note that Division does not apply. Say $X_1 \mapsto X$ and $X_2 \mapsto Y$. Show that the top X_2-degree coefficient of an algebraic relation is a multiple of X_1, and use this to set an induction on the top X_2-degree.