WORKSHEET #1.1: RINGS

EXAMPLE: The following are rings.

(1) Rings of numbers, like $\mathbb{Z}$ and $\mathbb{Z}[i] = \{a + bi \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$.

(2) Given a starting ring A, the polynomial ring in one indeterminate

$$A[X] := \{a_d X^d + \cdots + a_1 X + a_0 \mid d \geq 0, a_i \in A\},$$

or in a (finite or infinite![1]) set of indeterminates $A[X_1, \ldots, X_n]$, $A[X_\lambda \mid \lambda \in \Lambda]$.

(3) Given a starting ring A, the power series ring in one indeterminate

$$A[\![X]\!] := \left\{ \sum_{i \geq 0} a_i X^i \mid a_i \in A \right\},$$

or in a set of indeterminates $A[\![X_1, \ldots, X_n]\!]$.

(4) For a set X, $\mathrm{Fun}(X, \mathbb{R}) := \{\text{all functions } f : [0,1] \to \mathbb{R}\}$ with pointwise $+$ and $\times$.

(5) $\mathcal{C}([0,1]) := \{\text{continuous functions } f : [0,1] \to \mathbb{R}\}$ with pointwise $+$ and $\times$.

(6) $\mathcal{C}^\infty([0,1]) := \{\text{infinitely differentiable functions } f : [0,1] \to \mathbb{R}\}$ with pointwise $+$ and $\times$.

($\div$) Quotient rings: given a starting ring A and an ideal I, $R = A/I$.

($\times$) Product rings: given rings R and S, $R \times S = \{(r, s) \mid r \in R, s \in S\}$.

DEFINITION: An element x in a ring R is called a

- **unit** if x has an **inverse** $y \in R$ (i.e., $xy = 1$).
- **zerodivisor** if there is some $y \neq 0$ in R such that $xy = 0$.
- **nilpotent** if there is some $e \geq 0$ such that $x^e = 0$.
- **idempotent** if $x^2 = x$.

We also use the terms **nonunit, nonzerodivisor, nonnilpotent, nonidempotent** for the negations of the above. We say that a ring is **reduced** if it has no nonzero nilpotents.

(1) Warmup with units, zerodivisors, nilpotents, and idempotents.

(a) What are the implications between nilpotent, nonunit, and zerodivisor?

(b) What are the implications between reduced, field, and domain?

(c) What two elements of a ring are always idempotents? We call an idempotent **nontrivial** to mean that it is neither of these.

(d) If e is an idempotent, show that $e' := 1 - e$ is an idempotent[2] and $ee' = 0$.

(2) Elements in polynomial rings: Let $R = A[X_1, \ldots, X_n]$ a polynomial ring over a *domain* A.

(a) If $n = 1$, and $f, g \in R = A[X]$, briefly explain why the top degree[3] of fg equals the top degree of f plus the top degree of g. What if A is not a domain?

(b) Again if $n = 1$, briefly explain why $R = A[X]$ is a domain, and identify all of the units in R.

(c) Now for general n, show that R is a domain, and identify all of the units in R.

[1] Note: Even if the index set is infinite, by definition the elements of $A[X_\lambda \mid \lambda \in \Lambda]$ are finite sums of monomials (with coefficients in A) that each involve finitely many variables.

[2] We call e' the **complementary idempotent** to e.

[3] The **top degree** of $f = \sum a_i X^i$ is $\max\{k \mid a_k \neq 0\}$; we say **top coefficient** for a_k. We use the term top degree instead of degree for reasons that will come up later.

(3) Elements in power series rings: Let A be a ring.

- **(a)** Explain why the set of formal sums $\{\sum_{i\in\mathbb{Z}} a_i X_i \mid a_i \in A\}$ with arbitrary positive and negative exponents is *not* clearly a ring in the same way as $A[\![X]\!]$.
- **(b)** Given series $f, g \in A[\![X]\!]$, how much of f, g do you need to know to compute the X^3-coefficient of $f + g$? What about the X^3-coefficient of fg?
- **(c)** Find the first three coefficients for the inverse[4] of $f = 1 + 3X + 7X^2 + \cdots$ in $\mathbb{R}[\![X]\!]$.
- **(d)** Does "top degree" make sense in $A[\![X]\!]$? What about "bottom degree"?
- **(e)** Explain why[5] for a domain A, the power series ring $A[\![X_1, \ldots, X_n]\!]$ is also a domain.
- **(f)** Show[6] that $f \in A[\![X_1, \ldots, X_n]\!]$ is a unit if and only if the constant term of f is a unit.

(4) Elements in function rings.

- (a) For $R = \mathrm{Fun}([0,1], \mathbb{R})$,
  - (i) What are the nilpotents in R?
  - (ii) What are the units in R?
  - (iii) What are the idempotents in R?
  - (iv) What are the zerodivisors in R?
- (b) For $R = \mathcal{C}([0,1], \mathbb{R})$, $R = \mathcal{C}^\infty([0,1], \mathbb{R})$ same questions as above. When are there any/none?

(5) Product rings and idempotents.

- **(a)** Let R and S be rings, and $T = R \times S$. Show that $(1, 0)$ and $(0, 1)$ are nontrivial complementary idempotents in T.
- **(b)** Let T be a ring, and $e \in T$ a nontrivial idempotent, with $e' = 1 - e$. Explain why $Te = \{te \mid t \in T\}$ and Te' are rings with the same addition and multiplication as T. Why didn't I say "subring"?
- **(c)** Let T be a ring, and $e \in T$ a nontrivial idempotent, with $e' = 1 - e$. Show that $T \cong Te \times Te'$. Conclude that R has nontrivial idempotents if and only if R decomposes as a product.

(6) Elements in quotient rings:

- (a) Let K be a field, and $R = K[X,Y]/(X^2, XY)$. Find
  - a nonzero nilpotent in R
  - a zerodivisor in R that is not a nilpotent
  - a unit in R that is not equivalent to a constant polynomial
- (b) Find $n \in \mathbb{Z}$ such that
  - $[4] \in \mathbb{Z}/(n)$ is a unit
  - $[4] \in \mathbb{Z}/(n)$ is a nonzero nilpotent
  - $[4] \in \mathbb{Z}/(n)$ is a nonnilp. zerodivisor
  - $[4] \in \mathbb{Z}/(n)$ is a nontrivial idempotent

(7) More about elements.

- (a) Prove that a nilpotent plus a unit is always a unit.
- (b) Let A be an arbitrary ring, and $R = A[X]$. Characterize, in terms of their coefficients, which elements of R are units, and which elements are nilpotents.
- (c) Let A be an arbitrary ring, and $R = A[\![X]\!]$. Characterize, in terms of their coefficients, which elements of R are nilpotents.

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[4] It doesn't matter what the $\cdots$ are!

[5] You might want to start with the case $n = 1$.

[6] Hint: For $n = 1$, given $f = \sum_i a_i X^i$, construct $g = \sum_i b_i X^i$ by defining b_m recursively $b_0 = 1/a_0$ and that the X^m-coefficient of $(\sum_{i=0}^m a_i X^i)(\sum_{i=0}^m b_i X_i)$ is 0 for $m > 0$.