ASSIGNMENT #5: DUE FRIDAY, NOVEMBER 8 AT 7PM

This problem set is to be turned in by Canvas. You may reference any result or problem from our worksheets, unless it is the fact to be proven! You are encouraged to work with others, but you should understand everything you write. Please consult the class website for acceptable/unacceptable resources for the problem sets. You should use the techniques from this class and precursor classes to solve these problems, but not Commutative Algebra II or Homological Algebra.

(1) Topology of minimal primes:
  (a) Let $\mathfrak{p}$ be a minimal prime of R. Show that for any $a \in \mathfrak{p}$, there is some $u \notin \mathfrak{p}$ and $n \geq 1$ such that $ua^n = 0$.
  (b) Show that the set of minimal primes $\mathrm{Min}(R)$ with the induced topology from $\mathrm{Spec}(R)$ is Hausdorff.

(2) Let $\phi : R \to S$ be a ring homomorphism, and let $\phi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ be the induced map.
  (a) Show that the image of ϕ^* is contained in $V(\ker \phi)$.
  (b) Show that[1] any minimal prime of $\ker(\phi)$ is in the image of ϕ^*.
  (c) Show that the closure of the image of ϕ^* is $V(\ker \phi)$.
  (d) Find an example of a ring inclusion $\phi : R \subseteq S$ where the image of ϕ^* is not closed.

(3) Support of a module:
  (a) Given a finitely presented module L with $n \times m$ presentation matrix A, show that the support of L equals $V(I_n(A))$.
  (b) Let M be the $\mathbb{Z}$-module $\mathbb{Z}_2/\mathbb{Z}$. Compute the support of M and the annihilator of M, and show that the support of M is *not* equal to $V(\mathrm{ann}_{\mathbb{Z}}(M))$.
  (c) Let N be the $\mathbb{Z}$-module $\bigoplus_{p \text{ prime}} \mathbb{Z}/(p)$. Show that the support of N has infinitely many minimal elements.

(4) Use Macaulay2 to find the minimal primes and associated primes (via `minimalPrimes(I)` and `ass(I)`) of each of the following ideals. How many of them can you understand without Macaulay?
  (a) In $\mathbb{Q}[X, Y, U, V]$, the ideal $(X^2 - U^2, XY - UV, Y^2 - V^2)$.
  (b) In $\mathbb{Q}[X^4, X^3Y, X^2Y^2, XY^3, Y^4]$, the ideal (X^4).
  (c) In $\mathbb{Q}[X^4, X^3Y, XY^3, Y^4]$, the ideal (X^4).
  (d) In $\dfrac{\mathbb{Q}[U, V, W, X, Y, Z]}{I_2\left(\begin{bmatrix} U & V & W \\ X & Y & Z \end{bmatrix}\right)}$, the ideal $(U^3 + V^3 + X^3)$.
  (e) In $\mathbb{Q}\begin{bmatrix} X_{1,1} & X_{1,2} & Y_{1,1} & Y_{1,2} \\ X_{2,1} & X_{2,2} & Y_{2,1} & Y_{2,2} \end{bmatrix}$, the ideal given by the entries of the product of the matrices[2] $\begin{bmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{bmatrix}$ and $\begin{bmatrix} Y_{1,1} & Y_{1,2} \\ Y_{2,1} & Y_{2,2} \end{bmatrix}$.

---

[1] Hint: Consider the ring $\mathrm{Spec}\Big(\big(\phi(R \smallsetminus \mathfrak{p})\big)^{-1} S\Big)$. Relate its spectrum to a subset of $\mathrm{Spec}(S)$.

[2] You can do the multiplication by hand, or you can teach Macaulay a matrix with `matrix`. Try `viewHelp matrix` to find out how.