ASSIGNMENT #4: DUE FRIDAY, OCTOBER 25 AT 7PM

This problem set is to be turned in by Canvas. You may reference any result or problem from our worksheets, unless it is the fact to be proven! You are encouraged to work with others, but you should understand everything you write. Please consult the class website for acceptable/unacceptable resources for the problem sets. You should use the techniques from this class and precursor classes to solve these problems, but not Commutative Algebra II or Homological Algebra.

(1) Let $R = \mathbb{C}[X, Y, Z]$, and consider the ideals
$$I = (X^4 - Y^3, X^5 - Z^3, Y^5 - Z^4) \quad \text{and} \quad J = (X^3 - YZ, Y^2 - XZ, Z^2 - X^2Y).$$
Show that $I \subseteq J$ directly, compute $\mathcal{Z}(I)$, and use this show that $\sqrt{I} = \sqrt{J}$.

(2) Topological properties of $\operatorname{Spec}(R)$:
  (a) Show that $\operatorname{Spec}(R)$ is a Hausdorff space *only if* every prime ideal of R is maximal.
  (b) Show that for any ring R, $\operatorname{Spec}(R)$ is a compact topological space.
  (c) Let R be a polynomial ring in finitely many variables over an algebraically closed field. Show that for any closed subset $X \subseteq \operatorname{Spec}(R)$, the[1] subset $X \cap \operatorname{Max}(R)$ is dense in X.

(3) The KRULL INTERSECTION THEOREM states that for a Noetherian local ring $(R, \mathfrak{m})$ and a finitely generated module M, one has $\bigcap_{n \in \mathbb{N}} \mathfrak{m}^n M = 0$.
  (a) Prove the Krull Intersection Theorem.
  (b) Let $R = \mathcal{C}([0, 1], \mathbb{R})$, and $\mathfrak{m}$ be the maximal ideal of R consisting of all functions $f \in R$ such that $f(0) = 0$. Show that $\bigcap_{n \in \mathbb{N}} \mathfrak{m}^n = \mathfrak{m}$.

[1] where $\operatorname{Max}(R) \subseteq \operatorname{Spec}(R)$ denotes the set of maximal ideals of R.