ASSIGNMENT #5: DUE THURSDAY, OCTOBER 17 AT MIDNIGHT

This problem set is to be turned in on Canvas. You may reference any result or problem from our worksheets or lectures, unless it is the fact to be proven! You are encouraged to work with others, but you should understand everything you write. Please consult the class website for acceptable/unacceptable resources for the problem sets.

(1) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences. Either prove[1] or give a counterexample to each of the following:
  - (a) If $\{a_n^2\}_{n=1}^{\infty}$ is divergent, then $\{a_n\}_{n=1}^{\infty}$ is divergent.
  - (b) If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both diverge, then $\{a_n + b_n\}_{n=1}^{\infty}$ also diverges.
  - (c) If $\{a_n\}_{n=1}^{\infty}$ converges and $\{b_n\}_{n=1}^{\infty}$ diverges, then $\{a_n + b_n\}_{n=1}^{\infty}$ diverges.
  - (d) Suppose also that $b_n \neq 0$ for all $n \in \mathbb{N}$. If $\{b_n\}_{n=1}^{\infty}$ converges to 0, then $\left\{\dfrac{a_n}{b_n}\right\}_{n=1}^{\infty}$ diverges.

(2) Use the definition to prove that the sequence $\{-\sqrt{n}\}_{n=1}^{\infty}$ diverges to $-\infty$.

(3) Define a sequence $\{a_n\}_{n=1}^{\infty}$ recursively by $a_1 = 2$ and $a_{n+1} = \dfrac{a_n}{2} + \dfrac{1}{a_n}$.
  - (a) Use induction to prove that $a_n > 0$ for all $n \in \mathbb{N}$.
  - (b) Prove[2] that $a_n^2 \geq 2$ for all $n \in \mathbb{N}$.
  - (c) Prove[3] that the sequence is decreasing.
  - (d) Show that the sequence is convergent.
  - (e) Determine[4] to what value the sequence converges.

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[1] You may use our Theorem on Limits and Algebra whenever convenient, but make sure you are using something the Theorem says, and nothing it doesn't!

[2] Write $a_n^2 - 2$ in terms of a_{n-1} and factor the expression.

[3] Consider $a_{n+1} - a_n$ and use (b).

[4] If the sequence converges to L, explain why $L = \frac{L}{2} + \frac{1}{L}$, and solve for L.