ASSIGNMENT #10: DUE TUESDAY, DECEMBER 10 AT MIDNIGHT

This problem set is to be turned in on Canvas. You may reference any result or problem from our worksheets or lectures, unless it is the fact to be proven! You are encouraged to work with others, but you should understand everything you write. Please consult the class website for acceptable/unacceptable resources for the problem sets.

(1) Let f and g be functions defined on $\mathbb{R}$ and a a real number. Assume that f is differentiable at a and $f(a) = f'(a) = 0$.
  (a) Use the product rule to show that if g is differentiable at a, then $(fg)'(a) = 0$.
  (b) Show that[1] if g is continuous at a, then $(fg)'(a) = 0$.
  (c) Show that if g is *not* continuous at a, then fg may not be differentiable at a.

(2) Let $g : \mathbb{R} \to \mathbb{R}$ be the function given by the rule

$$g(x) = \begin{cases} x - x^2 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \notin \mathbb{Q}. \end{cases}$$

  (a) Show[2] that g is differentiable at $x = 0$ and $g'(0) = 1$.
  (b) Use a Theorem to explain why there is some $\delta > 0$ such that for all $x \in (0, \delta)$ we have $g(x) > g(0)$.
  (c) Find an explicit δ that works in the previous problem.
  (d) Show that there does not exist any $\delta > 0$ such that g is increasing on $(0, \delta)$.

(3) Let $p(x)$ be a polynomial function.
  (a) Use Rolle's Theorem to show that if $p(x)$ has n real roots, then $p'(x)$ has at least $n - 1$ real roots.
  (b) Use Rolle's Theorem and induction to show that if $p(x)$ is not a constant function, then the number of real roots of $p(x)$ is at most the degree of p.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$, $f(0) = 0$, and $f'(x) < 1$ for all $x \in \mathbb{R}$. Show that $f(x) < x$ for all $x > 0$.

[1]Note that the product rule does not apply! You might draw inspiration from the proof of the product rule.

[2]Suggestion: Reuse your work from an earlier assignment.