TRUE/FALSE. JUSTIFY AND/OR CORRECT.

(1) If (a, b) is a solution of the Pell's equation, then $a+b\sqrt{D}$ is a unit in $\mathbb{Z}[\sqrt{D}]$. T

(2) If $a+b\sqrt{D}$ is a unit in $\mathbb{Z}[\sqrt{D}]$, then (a, b) is a solution of the Pell's equation. F

(3) The norm function on $\mathbb{Z}[\sqrt{D}]$ satisfies the property $N(\alpha + \beta) = N(\alpha) + N(\beta)$. F

(4) For any integer D, the equation $x^2 - Dy^2 = 1$ has infinitely many solutions. T

(5) If D is a squarefree integer and m is any integer, then the equation $x^2 - Dy^2 = m$ has a solution. F

(6) If D is a squarefree integer and m is any integer, then the equation $x^2 - Dy^2 = m$ has either infinitely many solutions or no solutions. T

(7) The solutions to any Pell's equation forms a group. T

(8) A real number has a finite continued fraction expansion if and only if it is rational. T

(9) Every irrational number α can be approximated by infinitely many real numbers $\frac{p}{q}$ with $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$. T

(10) Every irrational number α can be approximated by infinitely many real numbers $\frac{p}{q}$ with $|\alpha - \frac{p}{q}| < \frac{1}{q^3}$. F

(11) The solution set in $\mathbb{R}^2$ to any polynomial $y^2 = x^3 + ax + b$ forms a group. F

(12) There exists an elliptic curve with exactly 9 rational points (not including ∞) that has a rational inflection point. F

COMPUTATIONS

(1) Find the inverse of [56] in $\mathbb{Z}_{89}$. [62]

(2) Compute $\left(\frac{-15}{43}\right)$. -1

(3) Compute $\left(\frac{37}{113}\right)$. -1

(4) Find the continued fraction expansion of $\frac{1}{\sqrt{3}}$. $[0;1,1,2,1,2,1,2,\ldots]$

(5) Given that $\sqrt{2} = [1; 2, 2, 2, 2, \ldots]$ and without using the decimal expansion of $\sqrt{2}$, determine which rational number with denominator less than 30 most closely approximates $\sqrt{2}$.
$41/29$

(6) Find the first three convergents for the continued fraction of $\sqrt[3]{2}$. $\frac{1}{1}, \frac{4}{3}, \frac{5}{4}, \frac{29}{23}$

(7) Find the first three positive integer solutions of the equation $x^2 = 7y^2 + 1$.
$(8,3), (127,48), (2024,765)$

(8) Give an expression for all solutions of the Pell's equation $x^2 - 12x^2 = 1$.
$(\pm x_k, \pm y_k), k \in \mathbb{N}$, where $x_k + y_k\sqrt{12} = (7 + 2\sqrt{12})^k$.

(9) Find the positive integer smallest solution of the equation $x^2 - 45y^2 = 1$. $(161,24)$

(10) For the points $P = (2, 2)$, $Q = (0, 2)$ in the real elliptic curve $\overline{E}$ given by $y^2 = x^3 - 4x + 4$, compute $P \star Q$ and $P \star P$. $P \star Q = (-2,2)$
$P \star P = Q$

(11) Compute the order of the point $P = ([3], [0])$ in the elliptic curve $\overline{E}_5$ given by $y^2 = x^3 + x$ over $\mathbb{Z}_5$. 2

(12) Compute the order of the point $P = ([3], [1])$ in the elliptic curve $\overline{E}_5$ given by $y^2 = x^3 + x + [1]$ over $\mathbb{Z}_5$. 7