QUADRATIC RESIDUES

DEFINITION: We say that an element $x \in \mathbb{Z}_n$ is a **square** or a **quadratic residue** if there is some $y \in \mathbb{Z}_n$ such that $y^2 = x$, and in this case, we call y a **square root** of x.

(1) Let n be an odd positive integer. Suppose that $[a]$ is a unit in $\mathbb{Z}_n$. Show that[1] the solutions x to the equation $[a]x^2 + [b]x + [c] = [0]$ in $\mathbb{Z}_n$ are exactly the elements of the form

$$x = \frac{-[b] + u}{[2a]} \quad \text{such that } u \text{ is a square root of } [b^2 - 4ac].$$

(2) Let p be an odd prime and $x \in \mathbb{Z}_p^\times$. Show that if x is a quadratic residue, then x has exactly two square roots $y \neq y'$, and for these roots, $y' = -y$.

(3) Let p be a prime number and g be a primitive root of $\mathbb{Z}_p$. Show that $[n] \in \mathbb{Z}_p^\times$ is a quadratic residue if and only if the index of $[n]$ with respect to g is even.

DEFINITION: Let p be an odd prime. For $r \in \mathbb{Z}$ not a multiple of p we define the **Legendre symbol** of r with respect to p as

$$\left(\frac{r}{p}\right) = \begin{cases} 1 & \text{if } [r] \text{ is a square in } \mathbb{Z}_p, \\ -1 & \text{if } [r] \text{ is a not square in } \mathbb{Z}_p. \end{cases}$$

THEOREM (EULER'S CRITERION): For p an odd prime and $r \in \mathbb{Z}$ not a multiple of p, we have

$$\left(\frac{r}{p}\right) \equiv r^{(p-1)/2} \pmod{p}.$$

THEOREM (QUADRATIC RECIPROCITY PART -1): If p is odd, then

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4} \\ -1 & \text{if } p \equiv 3 \pmod{4} \end{cases}.$$

PROPOSITION: Let p be an odd prime and a, b integers not divisible by p. Then

(1) $a \equiv b \pmod{p}$ implies that $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.

(2) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.

(3) $\left(\frac{a^2}{p}\right) = 1$.

[1] Hint: Complete the square!

(4) (a) Without using the Proposition above, explain why $\left(\frac{4}{p}\right) = 1$ for p an odd prime. Now explain why part (3) of the Proposition above is true in general.
(b) Use the Proposition above to explain the following: If a, b are not squares modulo p, then ab is a square modulo p.
(c) Use[2] the Proposition and Corollary above to determine how many solutions x to
$$[3]x^2 + [12]x - [2] = [0]$$
there are in $\mathbb{Z}_{43}$.

(5) Use problem #3 to prove Euler's criterion.

(6) Prove the proposition above.

(7) Use Euler's criterion to prove QR part -1 above.

(8) When n is not a prime...
(a) Does the conclusion of #4(b) hold if n is replaced by a general positive integer n instead of a prime p?
(b) Suppose that $n = pq$ for primes $p \neq q$. Show that a is a quadratic residue modulo n if and only if a is a quadratic residue modulo p and a quadratic residue modulo q.

[2] You might find it convenient to write $168 = 4 \cdot 42$.