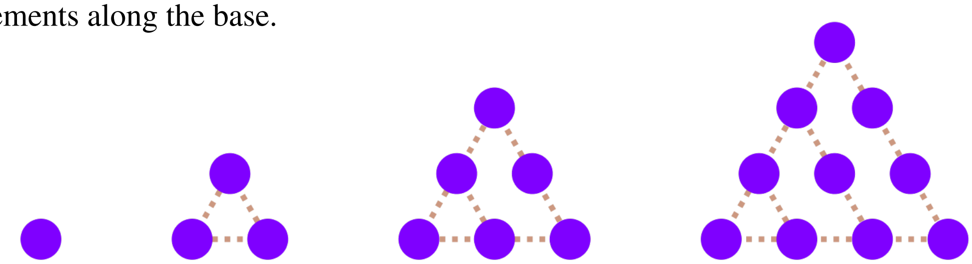
DEFINITION: A **triangular number** is a natural number T_n that counts the number of dots in a triangular array with n elements along the base.



$T_1 = 1$ $T_2 = 3$ $T_3 = 6$ $T_4 = 10$

(1) Explain why $T_n = 1+2+\cdots+n$. Then find[1] and prove a closed formula for the nth triangular number.

Going from T_n to T_{n+1}, we add one row of $n+1$ elements, so $T_{n+1} = T_n + (n+1)$, and the formula $T_n = 1+2+\cdots+n$ is then clear. For the second, we can write

$$\begin{aligned} T_n + T_n &= (1+2+\cdots+n) + (n+(n-1)+\cdots+1) \\ &= (n+1)+\cdots+(n+1) = n(n+1), \end{aligned}$$

so $T_n = \dfrac{n(n+1)}{2}$.

(2) In this problem we will classify all square-triangular numbers: numbers that are simultaneously triangular numbers and squares.

(a) Set $T_m = n^2$. Complete the square on the left-hand side, and clear denominators. Write x and y for the squares[2] appearing in the equation. What sort of equation in x and y do you get?

(b) Solve the equation in x and y. How is the integer solution set in the original equation in m and n related to the x and y equation?

(c) Use your work to write down the first four square-triangular numbers.

(a) We have $\dfrac{m(m+1)}{2} = n^2$, so $m^2+m = 2n^2$. Completing the square on the left gives

$$(m+\frac{1}{2})^2 - \frac{1}{4} = 2n^2,$$

and clearing denominators,

$$(2m+1)^2 - 1 = 8n^2.$$

Setting $x = 2m+1$ and $y = 2n$, we get the equation

$$x^2 - 2y^2 = 1,$$

a Pell's equation.

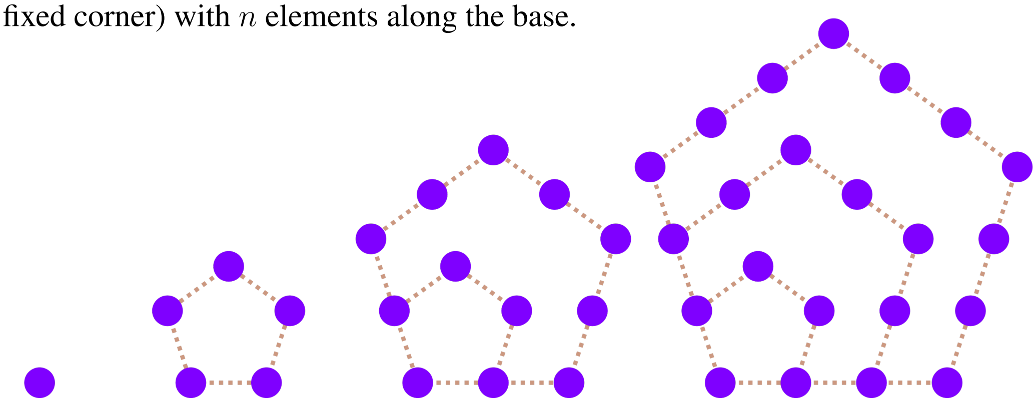
(b) The solutions (x,y) of $x^2 - 2y^2 = 1$ come from coefficients of $x + y\sqrt{2} = (3+2\sqrt{2})^k$ for $k \in \mathbb{N}$. Thus, if $T_m = n^2$ is a square-triangular number, $(2m+1, 2n) = (x,y)$ comes from the coefficients of $(3+2\sqrt{2})^k$ for $k \in \mathbb{N}$.
Conversely, given (x,y) such that $x+y\sqrt{2} = (3+2\sqrt{2})^k$, we can write $(x,y) = (2m+1, 2n)$ for some integers $m = \frac{x-1}{2}$, $n = \frac{y}{2}$ since x is odd and y is even: we can see this by induction,

[1]Hint: Write $T_n + T_n = (1+2+\cdots+n) + (n+(n-1)+\cdots+1)$.
[2]Suggestion: Write $8 = 2\cdot 2^2$ and include the 2 from 2^2 in y.

since if x is odd and y is even, then $(x + y\sqrt{2})(3 + 2\sqrt{2}) = (3x + 4y) + (3y + 2x)\sqrt{2}$ has $3x + 4y$ odd and $3y + 2x$ even.

(c) The first four solutions of the Pell's equation above are $(3, 2)$, $(17, 12)$, $(99, 70)$, $(577, 408)$. We then have $n = y/2$, and the actual number is $n^2 = y^2/4$. We get the numbers: $1, 36, 1225, 41616$.

DEFINITION: A **pentagonal number** is a natural number P_n that counts the number of dots in a pentagonal array (with a fixed corner) with n elements along the base.



$P_1 = 1$ $P_2 = 5$ $P_3 = 12$ $P_4 = 22$

(3) In this problem we will classify all square-pentagonal numbers: numbers that are simultaneously triangular numbers and squares.

(a) Find a formula for $P_m - P_{m-1}$. Use this and Problem (1) to give a closed formula for P_m.

(b) Set $P_m = n^2$. Complete the square on the left-hand side, and clear denominators. Write x and y for the squares appearing in the equation. What sort of equation in x and y do you get?

(c) Solve the equation in x and y. How is the integer solution set in the original equation in m and n related to the x and y equation? (Warning: This is more subtle than in the triangular case!)

(d) Use your work to write down the first three square-pentagonal numbers.

(a) $P_m - P_{m-1}$ corresponds to two sides of length m and one side of length $m - 2$, so $3m - 2$. Then we get

$$P_m = 3(1 + 2 + \cdots + m) - 2m = 3\frac{m(m+1)}{2} - 2m = \frac{3m^2 - m}{2}.$$

(b) We start with $\frac{3m^2 - m}{2} = n^2$, so $m^2 - \frac{m}{3} = \frac{2n^2}{3}$. Completing the square gives

$$(m - \frac{1}{6})^2 - \frac{1}{36} = \frac{2n^2}{3}$$

and clearing denominators yields

$$(6m - 1)^2 - 1 = 24n^2.$$

Set $x = 6m - 1$ and $y = 2n$ to get

$$x^2 - 6y^2 = 1,$$

a Pell's equation.

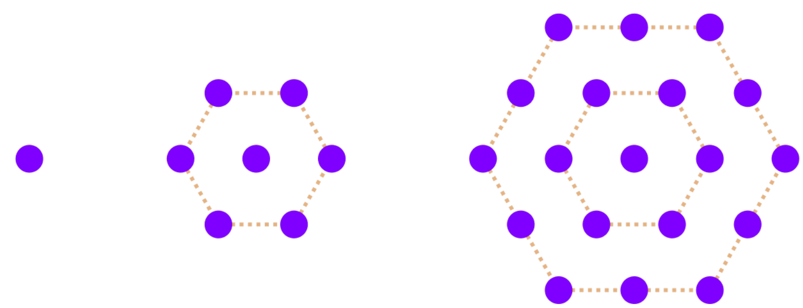
(c) We find $(x, y) = (5, 2)$ as the smallest positive solution to the Pell's equation above, so the solutions (x, y) of $x^2 - 6y^2 = 1$ come from coefficients of $x + y\sqrt{6} = (5 + 2\sqrt{6})^k$ for $k \in \mathbb{N}$.

Thus, if $P_m = n^2$ is a square-pentagonal number, $(6m-1, 2n) = (x, y)$ comes from the coefficients of $(5+2\sqrt{6})^k$ for $k \in \mathbb{N}$.

To determine which values of k yield an x with $x \equiv 5 \pmod 6$, and y even we compute that $(x+y\sqrt{6})(5+2\sqrt{6}) = (5x+12y)+(2x+5y)\sqrt{6}$. Since for $k=1$, y is even, by induction, we see that y is even for all k. Furthermore, if (x_k, y_k) is the kth solution, then $x_{k+1} \equiv 5x_k \pmod 6$. Since $x_1 \equiv 5 \pmod 6$, we determine that $x_k \equiv 5 \pmod 6$ exactly when k is *odd*.

(d) We compute the odd-indexed solutions to the Pell's equation $5+2\sqrt{6}$, $(5+2\sqrt{6})^3 = 485 + 198\sqrt{6}$, $(5+2\sqrt{6})^5 = 47525 + 19402\sqrt{6}$. Then we find that $n = y/2$, and the actual number we seek is n^2, so we get $1, 9801, 94109401$ as the square-pentagonal numbers.

DEFINITION: A **centered hexagonal number** is a number of the form is a natural number H_n that counts the number of dots in a hexagonal array (with a fixed center) with n elements along the base.



$H_1 = 1$ $H_2 = 7$ $H_3 = 19$

(4) Give a formula for all centered hexagonal numbers. Then give a formula for all square-(centered) hexagonal numbers, and list the first three of these.

We have $H_m - H_{m-1}$ is given by six sides with $m-1$ elements, so is $6(m-1)$. Using the formula from (1) we get that the mth centered hexagonal number is $3m^2 - 3m + 1$. Now we set $3m^2 - 3m + 1 = n^2$, and complete the square:

$$\begin{aligned} m^2 - m + \frac{1}{3} &= \frac{n^2}{3} \\ (m - \frac{1}{2})^2 - \frac{1}{4} + \frac{1}{3} &= \frac{n^2}{3} \\ 3(2m-1)^2 + 1 &= 4n^2, \end{aligned}$$

so setting $x = 2n$, $y = 2m-1$, we get the Pell's equation $x^2 - 3y^2 = 1$.

The solutions (x, y) to this Pell's equation are given by $x_k + y_k\sqrt{3} = (2+\sqrt{3})^k$. However, we only obtain integer solutions $(m, n) = (\frac{y+1}{2}, \frac{x}{2})$ to the original equation when x is even and y is odd. Multiplying out, we see that $x_{k+1} = 2x_k + 3y_k \equiv y_k \pmod 2$ and $y_{k+1} = x_k + 2y_k \equiv x_k \pmod 2$. Since x_1 is even and y_1 is odd, we have x_k even and y_k odd exactly when k is odd.

Then we take $n^2 = (x_k/2)^2$ for $k = 1, 3, 5$ to get $1, 169, 32761$.

(5) Find all numbers K that can be written in in the form

$$K = 1 + 2 + \cdots + (m-1) = (m+1) + (m+2) + \cdots + n$$

for some $m, n \in \mathbb{N}$. For example, the smallest such K is

$$15 = 1 + 2 + \cdots + 5 = 7 + 8.$$

In particular, find the first three such numbers.

Given m, n like so, we have

$$\begin{aligned} \frac{m(m-1)}{2} &= \frac{n(n+1)}{2} - \frac{m(m+1)}{2} \\ 2m^2 &= n^2 + n \\ 2m^2 &= \left(n + \frac{1}{2}\right)^2 - \frac{1}{4} \\ 2(2m)^2 &= (2n+1)^2 - 1 \end{aligned}$$

so setting $x = 2n + 1$ and $y = 2m$, we get solutions to Pell's equation $x^2 - 2y^2 = 1$. As above, we see that a solution (x, y) must have x odd and y even, so any solution to Pell's equation yields a solution (m, n) of the original; in particular,

$$K = \frac{\frac{y}{2}(\frac{y}{2} - 1)}{2} = \frac{y(y-2)}{8}.$$

From $y = 12, 70, 408$, we get $K = 15, 595, 20706$.

(6) Find all numbers K that can be written in in the form

$$K = 1 + 2 + \cdots + m = (m+1) + (m+2) + \cdots + n$$

for some $m, n \in \mathbb{N}$. For example, the two smallest such K are

$$3 = 1 + 2 = 3 \quad \text{and} \quad 105 = 1 + 2 + \cdots + 14 = 15 + 16 + \cdots + 20.$$