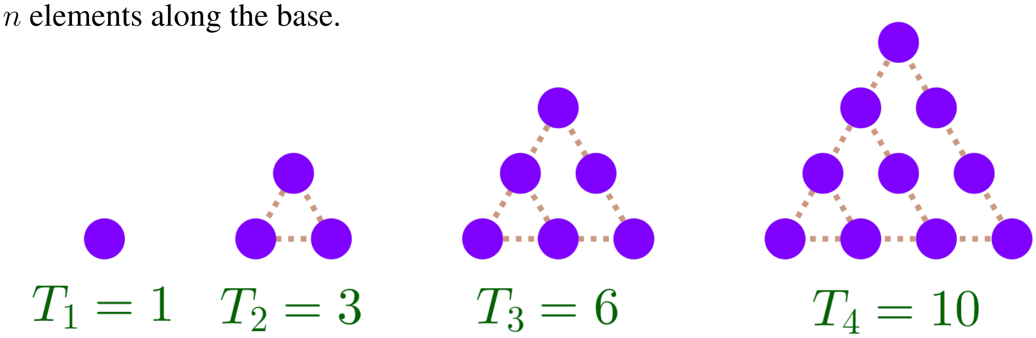
TRIANGULAR, SQUARE, PENTAGONAL, HEXAGONAL NUMBERS

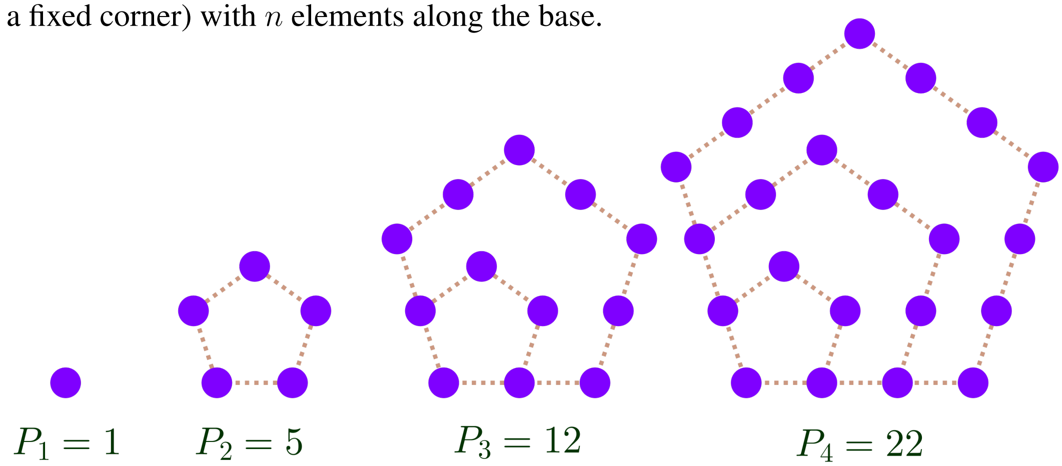
DEFINITION: A **triangular number** is a natural number T_n that counts the number of dots in a triangular array with n elements along the base.



(1) Explain why $T_n = 1+2+\cdots+n$. Then find[1] and prove a closed formula for the nth triangular number.

(2) In this problem we will classify all square-triangular numbers: numbers that are simultaneously triangular numbers and squares.
   (a) Set $T_m = n^2$. Complete the square on the left-hand side, and clear denominators. Write x and y for the squares[2] appearing in the equation. What sort of equation in x and y do you get?
   (b) Solve the equation in x and y. How is the integer solution set in the original equation in m and n related to the x and y equation?
   (c) Use your work to write down the first four square-triangular numbers.

DEFINITION: A **pentagonal number** is a natural number P_n that counts the number of dots in a pentagonal array (with a fixed corner) with n elements along the base.

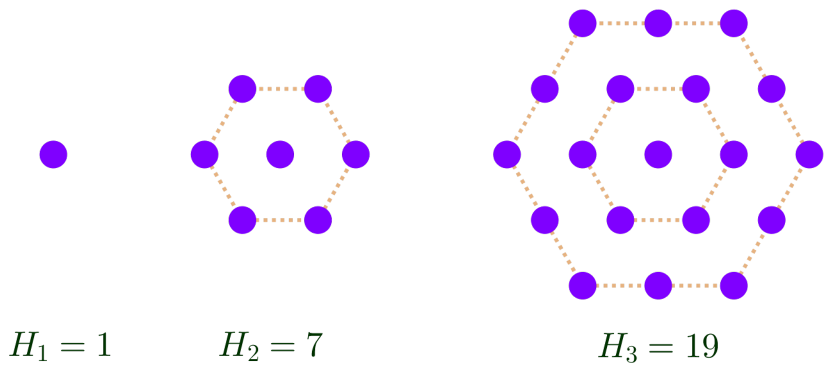


(3) In this problem we will classify all square-pentagonal numbers: numbers that are simultaneously triangular numbers and squares.
   (a) Find a formula for $P_m - P_{m-1}$. Use this and Problem (1) to give a closed formula for P_m.
   (b) Set $P_m = n^2$. Complete the square on the left-hand side, and clear denominators. Write x and y for the squares appearing in the equation. What sort of equation in x and y do you get?
   (c) Solve the equation in x and y. How is the integer solution set in the original equation in m and n related to the x and y equation? (Warning: This is more subtle than in the triangular case!)
   (d) Use your work to write down the first three square-pentagonal numbers.

[1] Hint: Write $T_n + T_n = (1+2+\cdots+n) + (n+(n-1)+\cdots+1)$.

[2] Suggestion: Write $8 = 2 \cdot 2^2$ and include the 2 from 2^2 in y.

DEFINITION: A **centered hexagonal number** is a number of the form is a natural number H_n that counts the number of dots in a hexagonal array (with a fixed center) with n elements along the base.



(4) Give a formula for all centered hexagonal numbers. Then give a formula for all square-(centered) hexagonal numbers, and list the first three of these.

(5) Find all numbers K that can be written in in the form

$$K = 1 + 2 + \cdots + (m-1) = (m+1) + (m+2) + \cdots + n$$

for some $m, n \in \mathbb{N}$. For example, the smallest such K is

$$15 = 1 + 2 + \cdots + 5 = 7 + 8.$$

In particular, find the first three such numbers.

(6) Find all numbers K that can be written in in the form

$$K = 1 + 2 + \cdots + m = (m+1) + (m+2) + \cdots + n$$

for some $m, n \in \mathbb{N}$. For example, the two smallest such K are

$$3 = 1 + 2 = 3 \quad \text{and} \quad 105 = 1 + 2 + \cdots + 14 = 15 + 16 + \cdots + 20.$$