THEOREM (EXISTENCE OF SOLUTIONS TO PELL'S EQUATION): Let D be a positive integer that is not a perfect square. Then the Pell's equation $x^2 - Dy^2 = 1$ has a positive solution.

THEOREM (SOLUTIONS TO PELL'S EQUATION ARE CONVERGENTS): Let D be a positive integer that is not a perfect square. For every positive solution (a, b) to the Pell's equation $x^2 - Dy^2 = 1$, there is some $k \in \mathbb{Z}_{\geq 0}$ such that the ratio $\frac{a}{b}$ is a convergent C_k of the continued fraction of $\sqrt{D}$.

THEOREM (GOOD APPROXIMATIONS ARE CONVERGENTS): Let r be an irrational real number. If p, q are integers with $q > 0$ such that $|r - \frac{p}{q}| < \frac{1}{2q^2}$, then there is some $k \in \mathbb{Z}_{\geq 0}$ such that $\frac{p}{q}$ is a convergent C_k of the continued fraction of r.

(1) Solving Pell's equation completely:
   (a) Given the theorems above, devise a method to find the smallest positive solution to the Pell's equation $x^2 - Dy^2 = 1$.
   (b) Apply your method for $D = 2$, $D = 3$, $D = 10$, and $D = 21$. Compare your results for $D = 2$ and $D = 3$ to what you found last time by trial and error.
   (c) Give a formula for all positive solutions to Pell's equation for $D = 10$ and $D = 21$.

(2) Prove the Theorem (Solutions to Pell's equation are convergents) using the Theorem (Good approximations are convergents).

(3) Proof of Theorem (Existence of solutions to Pell's equation):
   (a) Use Dirichlet's approximation theorem to show that there are infinitely many pairs of integers (x_i, y_i) such that $|x_i^2 - Dy_i^2| < 2\sqrt{D} + 1$.
   (b) Show that there is some integer m with $0 < |m| < 2\sqrt{D} + 1$ such that there are infinitely many pairs of integers (x_i, y_i) with $x_i^2 - Dy_i^2 = m$.
   (c) Show that there is some integer m with $|m| < 2\sqrt{D} + 1$ and $a, b \in \mathbb{Z}$ such that there are infinitely many pairs of integers (x_i, y_i) with

$$\begin{cases} x_i^2 - Dy_i^2 = m \\ x_i \equiv a \pmod{|m|} \\ y_i \equiv b \pmod{|m|} \end{cases}.$$

   (d) Given $i \neq j$ and x_i, x_j, y_i, y_j as in the previous part, show that $\frac{x_j + y_j\sqrt{D}}{x_i + y_i\sqrt{D}}$ is an element of $\mathbb{Z}[\sqrt{D}]$.
   (e) Complete the proof of the Theorem.

(4) Prove[1] Theorem (Good approximations are convergents).

[1]Hint: If not, we can assume $q_{k-1} < q < q_k$ for some k. In Problem set #5 problem #4, the same proof with $k - 1$ in place of k in parts (a)–(d) shows that, under the same hypotheses, $|qr - p| \geq |q_{k-1}r - p_{k-1}|$. Then show that $|\frac{p}{q} - \frac{p_{k-1}}{q_{k-1}}| < \frac{1}{qq_{k-1}}$.