DEFINITION: A **finite continued fraction** is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}$$

for some integers $a_0 \in \mathbb{Z}, a_1, \ldots, a_n \in \mathbb{Z}_{>0}$. We write $[a_0; a_1, \ldots, a_n]$ as shorthand for this.

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By a **continued fraction** we mean either an infinite or finite continued fraction. We call the numbers a_i the **partial quotients** in the continued fraction.

(1) Evaluating finite continued fractions:
   (a) Evaluate $2 + \dfrac{1}{13 + \frac{1}{2}}$.
   (b) Evaluate $[3; 2, 1, 4]$
   (c) Explain why every finite continued fraction evaluates to a rational number.

(a) $\frac{56}{27}$.
(b) $\frac{47}{14}$.
(c) A finite continued fraction is made out of integers from addition and division.

(2) Using the Euclidean algorithm to compute finite continued fractions:
   (a) What type of computation is the computation below?

$$\begin{aligned} 250 &= 2 \cdot 117 + 16 \\ 117 &= 7 \cdot 16 + 5 \\ 16 &= 3 \cdot 5 + 1 \\ 5 &= 5 \cdot 1 \end{aligned}$$

   (b) How does one obtain $\dfrac{250}{117} = 2 + \dfrac{1}{\frac{117}{16}}$ from the computation above?
   (c) Repeat (b) to obtain a finite continued fraction expansion for $\frac{250}{117}$.
   (d) Use the steps above to obtain a finite continued fraction expansion for $\frac{7}{5}$.
   (e) Use the steps above to obtain a finite continued fraction expansion for $\frac{39}{314}$.
   (f) What is the general formula for the continued fraction $[a_0; a_1, \ldots, a_n]$ for m/n in terms of the Euclidean algorithm?

(a) Euclidean algorithm.
(b) Divide the first line by 117 and flip the last fraction.
(c) $\frac{250}{117} = 2 + \frac{1}{7 + \frac{1}{3 + \frac{1}{5}}}$.
(d) $\frac{7}{5} = 1 + \frac{1}{2 + \frac{1}{2}}$.
(e) $\frac{39}{314} = \frac{1}{8 + \frac{1}{19 + \frac{1}{2}}}$.
(f) The a_i's are just the quotients in the Euclidean algorithm.

(3) Euclidean algorithm and continued fraction algorithm:

(a) In the computation from (2a) above, check that

$$2 = \left\lfloor \frac{250}{117} \right\rfloor \text{ and that } \frac{117}{16} = \left(\frac{250}{117} - \left\lfloor \frac{250}{117} \right\rfloor \right)^{-1}.$$

(b) More generally, in the Euclidean algorithm

$$\begin{array}{rcccccl} \vdots & & \vdots & & \vdots & & \vdots \\ u_i & = & q_i & \cdot & v_i & + & r_i \qquad\qquad (u_{i+1} = v_i) \\ u_{i+1} & = & q_{i+1} & \cdot & v_{i+1} & + & r_{i+1} \qquad\qquad (v_{i+1} = r_i) \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

show that

$$q_i = \left\lfloor \frac{u_i}{v_i} \right\rfloor \text{ and } \frac{u_{i+1}}{v_{i+1}} = \left(\frac{u_i}{v_i} - \left\lfloor \frac{u_i}{v_i} \right\rfloor \right)^{-1}.$$

(a) ✓

(b) The formula for q_i is the general formula in the division algorithm (since $u_i/v_i - 1 < \lfloor u_i/v_i \rfloor \leq u_i/v_i$ implies $v_i > u_i - \lfloor u_i/v_i \rfloor v_i \geq 0$.) We then have

$$\frac{u_{i+1}}{v_{i+1}} = \frac{v_i}{r_i} = \frac{v_i}{u_i - \lfloor \frac{u_i}{v_i} \rfloor v_i} = \frac{1}{\frac{u_i}{v_i} - \lfloor \frac{u_i}{v_i} \rfloor}.$$

DEFINITION: Given an infinite continued fraction $[a_0; a_1, a_2, \ldots]$, the k-th **convergent** of the continued fraction is the value C_k of the finite continued fraction $[a_0; a_1, \ldots, a_k]$.

THEOREM (CONVERGENCE OF CONTINUED FRACTIONS): Every infinite continued fraction converges to a real number; i.e., for any $[a_0; a_1, a_2, a_3, \ldots]$ with $a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots \in \mathbb{Z}_{>0}$, the sequence of convergents $C_1, C_2, C_3, \ldots$ converges. We call this limit the value of the infinite continued fraction.

CONTINUED FRACTION ALGORITHM: Given a real number r,

(I) Start with $\beta_0 := r$ and $n := 0$.

(II) Set $a_n := \lfloor \beta_n \rfloor$.

(III) If $a_n = \beta_n$, STOP; the continued fraction is $[a_0; a_1, \ldots, a_n]$.
Else, set $\beta_{n+1} := (\beta_n - a_n)^{-1}$, and return to Step (II).

If the algorithm does not terminate, the continued fraction is $[a_0; a_1, a_2, \ldots]$.

THEOREM (CORRECTNESS OF CONTINUED FRACTION ALGORITHM): For any real number r, the continued fraction obtained from the Continued Fraction Algorithm with input r converges to r.

PROPOSITION: Let r be a real number. The Continued Fraction Algorithm with input r terminates in finitely many steps if and only if r is rational.

DIRICHLET APPROXIMATION THEOREM: Let $r = [a_0; a_1, a_2, a_3, \ldots]$ be a real number. Then for every convergent $C_k = \frac{p_k}{q_k}$ (in lowest terms), we have $\left| r - \frac{p_k}{q_k} \right| < \frac{1}{q_k^2}$.

In particular, if r is irrational, there are infinitely many rational numbers $\frac{p}{q}$ such that $\left| r - \frac{p}{q} \right| < \frac{1}{q^2}$.

(4) Use the continued fraction algorithm to find the first four ($n \leq 3$) partial quotients and convergents for $\sqrt{2}$, and π. Can you find the whole continued fraction for either of these?

> $\sqrt{2} = [1; 2, 2, 2, \dots]$ and 2's forever, since $\beta_i = \sqrt{2} + 1$ for all $i > 0$, with $C_0, C_1, C_2, C_3 = 1, 3/2, 7/5, 12/5$. $\pi = [3; 7, 15, 1, \dots]$ and a mysterious pattern, with $C_0, C_1, C_2, C_3 = 3, 22/7, 333/106, 355/113$.

(5) Find[1] the value of the continued fraction $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}$.

> We have $L = 1 + 1/L$, so $L^2 = L + 1$. This has two roots $\frac{1 \pm \sqrt{5}}{2}$. Since $L > 0$, we must have $L = \frac{1+\sqrt{5}}{2}$, the golden ratio.

(6) Continued fraction algorithm and rational numbers.
  (a) Explain why the continued fraction algorithm just creates a continued fraction in the same way the Euclidean algorithm does as we did in problem (2).
  (b) Explain why the Proposition above is true.

> (a) This was the point of problem (3).
> (b) If the algorithm terminates, then r has a finite continued fraction, and hence is rational. Conversely, if r is rational, the continued fraction algorithm follows the Euclidean algorithm and after finitely many steps returns a finite continued fraction.

(7) Dirichlet Approximation Theorem.
  (a) Let r be any real number. Explain why for *any* positive integer q, there is some integer p such that $|r - \frac{p}{q}| < \frac{1}{q}$. Conclude that $|r - \frac{p}{q}| < \frac{1}{q}$ is "not very impressive".
  (b) For $r = \sqrt{2}$, find all rational numbers p/q with $|r - \frac{p}{q}| < \frac{1}{q^2}$ with $q \leq 6$ and compare to the list of convergents C_0, C_1, C_2. What about $|r - \frac{p}{q}| < \frac{1}{2q^2}$? Conclude that $|r - \frac{p}{q}| < \frac{1}{q^2}$ is "pretty impressive".
  (c) Discuss $\pi \approx \frac{22}{7}$ in the context of the results above. Give a better approximation.

> (a) Set $p = \lfloor r/q \rfloor$.
> (b) For the first, we just have C_0, C_1, C_2 along with $\frac{2}{1}$ and $\frac{4}{3}$. For the second, just C_0, C_1, C_2. We are impressed.
> (c) This is a good approximation in the sense of Dirichlet Approximation Theroem, since it comes from the continued fraction. $\pi \approx 355/113$ is a very good approximation.

> PROPOSITION: Let $[a_0; a_1, a_2, \dots]$ be a continued fraction. Set
> $$\begin{aligned} p_0 &:= a_0, & p_1 &:= a_0 a_1 + 1, & p_k &:= a_k p_{k-1} + p_{k-2} \\ q_0 &:= 1, & q_1 &:= a_1, & q_k &:= a_k q_{k-1} + q_{k-2}. \end{aligned}$$
> Then,
> (1) $C_k = \dfrac{p_k}{q_k}$ for all $k \geq 0$, and
> (2) $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$ for all $k \geq 1$.

[1]Hint: This limit has a value L. Find an equation that L satisfies by recognizing L as a smaller piece of this continued fraction.

(8) Proof of convergence Theorem and Dirichlet Approximation Theorem.

(a) Use the Proposition above to show that $C_k - C_{k-1} = \dfrac{(-1)^{k-1}}{q_k q_{k-1}}$ for all $k \geq 1$.

(b) Use the Proposition above to show that $C_k - C_{k-2} = \dfrac{(-1)^k a_k}{q_k q_{k-2}}$ for all $k \geq 2$.

(c) Use (8b) to show that the sequence $C_0, C_2, C_4, \ldots$ is increasing, that the sequence $C_1, C_3, C_5, \ldots$ is decreasing; use (8a) to show that $C_{2k} < C_{2\ell+1}$ for all k, ℓ. Deduce that $\lim_{k\to\infty} C_{2k} = \sup\{C_{2k} \mid k \in \mathbb{N}\}$ and $\lim_{\ell\to\infty} C_{2\ell+1} = \inf\{C_{2\ell+1} \mid \ell \in \mathbb{N}\}$ both exist.

(d) Use (8a) to show that $\sup\{C_{2k} \mid k \in \mathbb{N}\} = \inf\{C_{2\ell+1} \mid \ell \in \mathbb{N}\}$, and hence that $\lim_{n\to\infty} C_n$ exists and is equal to both of these values. Thus, every continued fraction converges.

(e) Suppose that β is the value of our continued fraction. Use (8d) to show that $|\beta - C_n| \leq |C_{n+1} - C_n|$, and use (8a) to deduce Dirichlet's Approximation.

(a)

$$C_k - C_{k-1} = \frac{p_k}{q_k} - p_{k-1}q_{k-1} = \frac{p_k q_{k-1} - p_{k-1} q_k}{q_k q_{k-1}} = \frac{(-1)^{k-1}}{q_k q_{k-1}}$$

(b)

$$C_k - C_{k-2} = C_k - C_{k-1} + C_{k-1} - C_{k-2} = \frac{(-1)^{k-1}}{q_k q_{k-1}} + \frac{(-1)^{k-2}}{q_{k-1} q_{k-2}}$$

$$= (-1)^k \frac{-q_{k-2} + q_k}{q_k q_{k-1} q_{k-2}} = \frac{(-1)^k a_k q_{k-1}}{q_k q_{k-1} q_{k-2}} = \frac{(-1)^k a_k}{q_k q_{k-2}}$$

(c) From (8b), we have $C_k - C_{k-2} > 0$ (so $C_k > C_{k-2}$) if k is even and $C_k - C_{k-2} < 0$ (so $C_k < C_{k-2}$) if k is odd. Thus, the sequence $C_0, C_2, C_4, \ldots$ is increasing and the sequence $C_1, C_3, C_5, \ldots$ is decreasing. By (8a), $C_{2\ell+1} - C_{2\ell} > 0$, so $C_{2\ell+1} > C_{2\ell}$; if $\ell \leq k$, then $C_{2\ell+1} > C_{2\ell} > C_{2k}$; if $\ell \geq k$, then $C_{2\ell+1} > C_{2k+1} > C_{2k}$. Then the sequence $(C_{2k})_{k=1}^{\infty}$ is increasing and bounded above (by, e.g., C_1), and the sequence $(C_{2\ell+1})_{\ell=1}^{\infty}$ decreasing and bounded below (by, e.g., C_0). By the monotone convergence theorem, these sequences converge to their sup and inf, respectively.

(d) Suppose $\sup\{C_{2k}\} < \inf\{C_{2\ell+1}\}$, and left $\delta = \inf\{C_{2\ell+1}\} - \sup\{C_{2k}\}$. Let $2n$ be an even number larger than $1/\delta$. Then

$$C_{2n} < \sup\{C_{2k}\} < \inf\{C_{2\ell+1}\} < C_{2n+1}$$

implies $C_{2n+1} - C_{2n} > 1/(2n)$, but we also have $C_{2n+1} - C_{2n} = 1/(q_{2n} q_{2n-1})$. Since $q_{2n} > 2n$, this is a contradiction. It follows that the sequence of convergents converges.

(e) If n is even, then we have $C_n < \sup\{C_{2k}\} = \beta = \inf\{C_{2\ell+1}\} < C_{n+1}$, and if n is odd, we have $C_{n+1} < \sup\{C_{2k}\} = \beta = \inf\{C_{2\ell+1}\} < C_n$. This shows that $|\beta - C_n| \leq |C_{n+1} - C_n|$. Then from (8a), $|C_{n+1} - C_n| = 1/(q_n q_{n+1}) < 1/q_n^2$.

(9) Prove the Proposition above.

We prove (1) by induction on k. We need two base cases, $k = 0$ and $k = 1$. For those, we have $[a_0;] = a_0/1$, and $[a_0; a_1] = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1}$. Now for the inductive step, suppose this holds for continued fractions of length at most k. Then we can write $C_{k+1} = [a_0; a_1, \ldots, a_k, a_{k+1}] = [a_0; a_1, \ldots, a_k']$, where $a_k' = a_k + 1/a_{k+1}$. We apply the IH to the latter continued fraction:

$$C_{k+1} = \frac{a_k' p_{k-1} + p_{k-2}}{a_k' q_{k-1} + q_{k-2}} = \frac{(a_k + 1/a_{k+1}) p_{k-1} + p_{k-2}}{(a_k + 1/a_{k+1}) q_{k-1} + q_{k-2}}$$

$$= \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}} = \frac{a_{k+1} p_k + p_{k-1}}{a_{k+1} q_k + q_{k-1}} = \frac{p_{k+1}}{q_{k+1}},$$

completing the induction.

We prove (2) by induction too. For $k = 1$, we get

$$p_1 q_0 - p_0 q_1 = (a_0 a_1 + 1) \cdot 1 - a_0 a_1 = 1.$$

Assume the formula holds for k, so

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}.$$

Then

$$\begin{aligned} p_{k+1} q_k - p_k q_{k+1} &= (a_{k+1} p_k + p_{k-1}) q_k - p_k (a_{k+1} q_k + q_{k-1}) \\ &= p_{k-1} q_k - p_k q_{k-1} = -(-1)^{k-1} = (-1)^k, \end{aligned}$$

completing the inductive step.

(10) Proof of Correctness of Continued Fraction Algorithm:

If r is rational, the algorithm terminates and returns r, so we can assume that r is irrational and that the algorithm does not terminate. Given r, let $a_0, a_1, a_2, a_3, \ldots$ and $\beta_0, \beta_1, \beta_2, \ldots$ be the sequences arising from the continued fraction algorithm.

(a) Explain why $r = [a_0; a_1, \ldots, a_k, \beta_{k+1}]$. (Note, β_{k+1} is not an integer, but we can plug it into a finite continued fraction anyway.)

(b) Explain why $r = \dfrac{\beta_{k+1} p_k + p_{k-1}}{\beta_{k+1} q_k + q_{k-1}}$ where p_k, q_k, where p_k, q_k are the numbers coming from the continued fraction (with an irrational number snuck in) $[a_0; a_1, \ldots, a_k, \beta_{k+1}]$ as in the Proposition above.

(c) Show that $|r - C_k| < \dfrac{1}{q_k q_{k+1}}$ for all $k \geq 1$ and deduce the result.

(a) We argue by induction on k. Since $\beta_0 = r$ and $[a_0;]$ means a_0, the case $k = 0$ holds. If $r = [a_0; a_1, \ldots, a_k, \beta_{k+1}]$, then by definition $\beta_{k+2} = 1/(\beta_{k+1} - a_{k+1})$, so $\beta_{k+1} = a_{k+1} + \frac{1}{\beta_{k+2}}$. Plugging this into the continued fraction setup, $r = [a_0; a_1, \ldots, a_k, a_{k+1}, \beta_{k+2}]$. This completes the induction.

(b) The same proof as the Proposition works.

(c)

$$\begin{aligned} r - C_k &= \frac{\beta_{k+1} p_k + p_{k-1}}{\beta_{k+1} q_k + q_{k-1}} - \frac{p_k}{q_k} \\ &= \frac{\beta_{k+1} p_k q_k + p_{k-1} q_k - p_k \beta_{k+1} q_k - p_k q_{k-1}}{(\beta_{k+1} q_k + q_{k-1}) q_k} \\ &= \frac{p_{k-1} q_k - p_k q_{k-1}}{(\beta_{k+1} q_k + q_{k-1}) q_k} = \frac{(-1)^k}{(\beta_{k+1} q_k + q_{k-1}) q_k} \end{aligned}$$

Since $\beta_{k+1} > a_{k+1}$, we have

$$\beta_{k+1} q_k + q_{k-1} > a_{k+1} q_k + q_{k-1} = q_{k+1}$$

so

$$|r - C_k| < \frac{1}{q_{k+1} q_k} < \frac{1}{q_k^2}.$$

(11) Prove the following theorem, which basically says that the convergents are the *best* approximations of a rational number.

THEOREM: Let r be a real number, $C_k = \frac{p_k}{q_k}$ be the k-th convergent of r, and $\frac{p}{q} \neq r$ be a rational number. If $q \leq q_k$, then $\left| r - \dfrac{p}{q} \right| \geq \left| r - \dfrac{p_k}{q_k} \right|$.