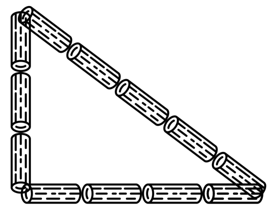
DEFINITION: A triple (a, b, c) of natural numbers is a **Pythagoran triple** if they form the side lengths of a right triangle, where c is the length of the hypotenuse.



$(3, 4, 5)$ is a Pythagorean triple.

Our goal today is to find all Pythagoran triples. We will use a couple of tools that whose relevance might not be clear at first:

FUNDAMENTAL THEOREM OF ARITHMETIC: Every natural number $n \geq 1$ can be written as a product of prime numbers:

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}.$$

This expression is unique up to reordering. □

We call the number e_i the **multiplicity** of the prime p_i in the prime factorization of n.

DEFINITION: Let m, n be integers and $K \geq 1$ be a natural number. We say that m **is congruent to** n **modulo** K, written as $m \equiv n \pmod{K}$, if $m - n$ is a multiple of K.

THEOREM: Let n be an integer and $K \geq 1$ a natural number. Then n is congruent to exactly one nonnegative integer between 0 and $K-1$: this number is the "remainder" when you divide n by K. □

PROPOSITION: Let m, m', n, n' and K be natural numbers. Suppose that

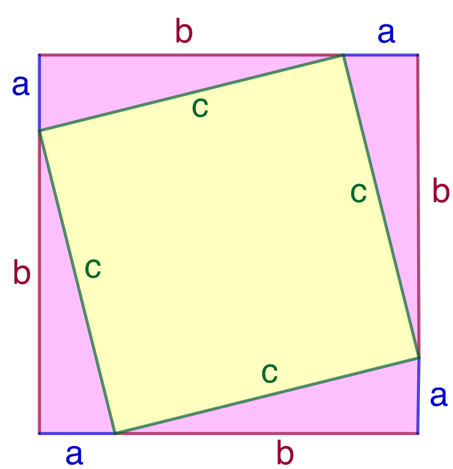
$$m \equiv m' \pmod{K} \quad \text{and} \quad n \equiv n' \pmod{K}.$$

Then

$$m + n \equiv m' + n' \pmod{K} \quad \text{and} \quad mn \equiv m'n' \pmod{K}.$$

□

(1) Without writing too much, use the picture below to deduce the PYTHAGOREM THOREM: If a, b, c are the side lengths of a right triangle, where c is the length of the hypotenuse, then $a^2 + b^2 = c^2$.



(2) Creating Pythagorean triples from others:
   (a) Show that if (a, b, c) is a Pythagorean triple and d is a natural number, then (da, db, dc) is a Pythagorean triple. Deduce that there are infinitely many Pythagorean triples.
   (b) Show that if (a, b, c) is a Pythagorean triple and d is a common factor of a, b, and c, then $(a/d, b/d, c/d)$ is a Pythagorean triple.

> DEFINITION: A triple (a, b, c) of natural numbers is a **primitive Pythagoran triple (PPT)** if $a^2 + b^2 = c^2$, and there is no common factor of a, b, c greater than 1; equivalently, a, b, c have no common prime factor.
>
> Based on (1) and (2), finding all Pythagorean triples boils down to finding all PPTs.

(3) Let a be a natural number. Show that if a is even, then $a^2 \equiv 0 \pmod 4$, and if a is odd, then $a^2 \equiv 1 \pmod 4$.

(4) Suppose that (a, b, c) is a Pythagorean triple. We want to examine the parity (even vs. odd) of the numbers a, b, c.
   (a) Suppose that a and b are both even. Show that c is even too. Deduce that there are no PPTs with a and b both even.
   (b) Suppose now that a and b are both odd. Consider the equation $a^2 + b^2 = c^2$ modulo 4, and use the problem (3) to get a contradiction.
   (c) Conclude that if (a, b, c) is a PPT, then one of a, b is odd, and the other is even, and that c is odd.

(5) Let m and n be natural numbers.
   (a) Show that n is a perfect square if and only if the multiplicity of each prime in its prime factorization is even.
   (b) Suppose that m and n have no common prime factors. Show that if mn is a perfect square, then m and n are both perfect squares.

(6) Consider a PPT (a, b, c). Following (4c), without loss of generality we can assume that a is odd and b is even. Rewrite the equation $a^2 + b^2 = c^2$ as $a^2 = c^2 - b^2$.
   (a) By definition, there is no prime factor common to all three of a, b, and c. Show that there is no prime factor common to just b and c.
   (b) Factor $c^2 - b^2$ as $(c - b)(c + b)$. Show that[1] there is no prime factor common to $c - b$ and $c + b$.
   (c) Show that $c - b$ and $c + b$ are perfect squares.
   (d) Show[2] that any PPT can be written in the form

$$(a, b, c) = \left(st, \frac{s^2 - t^2}{2}, \frac{s^2 + t^2}{2} \right)$$

   for some odd integers $s > t \geq 1$ with no common factors.
   (e) Check the other direction: show that any triple of the form $(st, \frac{s^2-t^2}{2}, \frac{s^2+t^2}{2})$, where $s > t \geq 1$ are odd integers with no common factors, is a PPT.

[1]Hint: If there is a (prime) number that divides these, it divides their sum and difference too.
[2]Hint: Start with writing $c + b = s^2$, $c - b = t^2$ and solve for a, b, c.

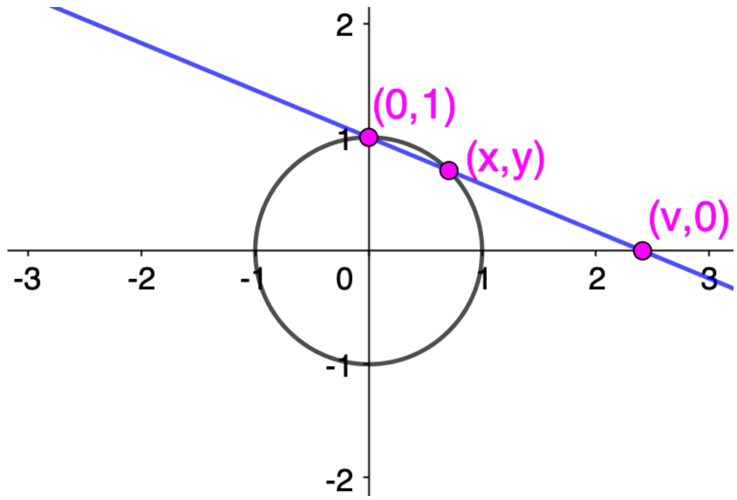
You have proven the following:

THEOREM: The set of primitive Pythagorean triples (a, b, c) with a odd is given by the formula

$$a = st, \quad b = \frac{s^2 - t^2}{2}, \quad c = \frac{s^2 + t^2}{2},$$

where $s > t \geq 1$ are odd integers with no common factors.

These mysterious formulas have a geometric explanation.



(7) (a) Show that if (a, b, c) is a Pythagorean triple, then $\left(\frac{a}{c}, \frac{b}{c}\right)$ is a point on the circle with positive rational coordinates, and vice versa.

(b) Given a rational number $v > 1$, the line L through $(0, 1)$ and $(v, 0)$ intersects the unit circle in two points (one of which is $(0, 1)$). As a first step towards finding this point, find an equation for L.

(c) Use the equation you found in (7b) and the equation for the unit circle to solve for x and y in terms of v.

(d) Use (b) to solve for v in terms of x and y and this to show that if x and y are rational, then v is rational.

Conclude the following theorem:

THEOREM: The set of points on the unit circle $x^2 + y^2 = 1$ with positive rational coordinates is given by the formula

$$(x, y) = \left(\frac{2v}{v^2 + 1}, \frac{v^2 - 1}{v^2 + 1}\right)$$

where v ranges through rational numbers greater than one.

(e) Take the expressions for x and y from the Theorem above in terms of v, and plug in $v = s/t$ and simplify each expression for x and y into a single fraction.

(f) Plug these expressions back into $x^2 + y^2 = 1$, clear denominators, and divide through by 4. What do you notice?

(8) Use similar techniques[3] to find rational points on:
   (a) The circle $x^2 + y^2 = 2$.
   (b) The hyperbola $x^2 - y^2 = 1$.
   (c) The hyperbola $x^2 - 2y^2 = 1$.
   (d) The circle $x^2 + y^2 = 3$.

(9) Use this to find integer solutions (a, b, c) to the equations:
   (a) The circle $a^2 + b^2 = 2c^2$.
   (b) The hyperbola $a^2 - b^2 = c^2$.
   (c) The hyperbola $a^2 - 2b^2 = c^2$.
   (d) The circle $a^2 + b^2 = 3c^2$.

   Are these all of the integer solutions?

Key Points:

- Using the Fundamental Theorem of Arithmetic for basic divisibility arguments.
- Definition of congruence, and using congruences to rule out solutions of equations.
- Using geometry to find rational points.

[3] Hint: You many have to change your starting point and/or target line. You might find it useful to take new coordinates in which your starting point is the origin, i.e., $x' = x - a$, $y' = y - b$ if your starting point is (a, b).