Math 445 — Problem Set #7
Due: Tuesday, November 14 by 7 pm, on Canvas

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times. Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Let E be a real elliptic curve. Recall that a point $P \in \overline{E}$ has order 2 if and only if P has a vertical tangent line. Prove[1] that every point of order 2 in $\overline{E}$ is a point on the x-axis, and that $\overline{E}$ has at most three points of order 2.

(2) Find all powers $P, 2P, 3P, \ldots$ of the point $P = (3, 8)$ in $E : y^2 = x^3 - 43x + 166$. You can, and may want to, use a computer graphing system to start by computing small powers.

(3) In this problem, we will prove that the elliptic curve $E : y^2 = x^3 + 7$ has no integer solutions.
- (a) Suppose that (a, b) is an integer solution. Show that a must be odd.
- (b) Show that $b^2 + 1 = (a + 2)((a - 1)^2 + 3)$.
- (c) Show that there exists a prime $q \equiv 3 \pmod 4$ that divides the integer in (b), and obtain a contradiction.

[1] Use calculus.