

Math 445 — Problem Set #6
Due: Friday, November 3 by 7 pm, on Canvas

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. If you do work with others, I ask that you write something along the top like “I collaborated with Steven Smale on problems 1 and 3”. If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times. Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

- (1) Use the methods from class to give a formula¹ for all solutions of the Pell's equation

$$x^2 - 13y^2 = 1.$$

We use the continued fraction algorithm:

$$\begin{aligned} \sqrt{13} &= 3 + (\sqrt{13} - 3) = 3 + \frac{1}{\left(\frac{1}{\sqrt{13}-3}\right)} = 3 + \frac{1}{\frac{\sqrt{13}+3}{4}} = 3 + \frac{1}{1 + \frac{\sqrt{13}-1}{4}} = 3 + \frac{1}{1 + \frac{1}{\left(\frac{4}{\sqrt{13}-1}\right)}} \\ &= 3 + \frac{1}{1 + \frac{1}{\left(\frac{4(\sqrt{13}+1)}{12}\right)}} = 3 + \frac{1}{1 + \frac{1}{\left(\frac{\sqrt{13}+1}{3}\right)}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{\sqrt{13}-2}{3}}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\left(\frac{3}{\sqrt{13}-2}\right)}}} \\ &= 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\left(\frac{3(\sqrt{13}+2)}{9}\right)}}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\left(\frac{\sqrt{13}+2}{3}\right)}}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{\sqrt{13}-1}{3}}}} \\ &= 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\left(\frac{3}{\sqrt{13}-1}\right)}}}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\left(\frac{3(\sqrt{13}+1)}{12}\right)}}}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\left(\frac{\sqrt{13}+1}{4}\right)}}}} \\ &= 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{\sqrt{13}-3}{4}}}}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\left(\frac{4(\sqrt{13}+3)}{4}\right)}}}}} \\ &= 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\sqrt{13}+3}}}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{6 + (\sqrt{13}-3)}}}} \end{aligned}$$

and since $\sqrt{13} - 3$ appears as a remainder in the first step, the continued fraction must start repeating. That is, $\sqrt{13} = [3; 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, \dots]$.

We use this to generate a list of convergents $\frac{a}{b}$:

$$\frac{3}{1}, \frac{4}{1}, \frac{7}{2}, \frac{11}{3}, \frac{18}{5}, \frac{119}{33}, \frac{137}{38}, \frac{256}{71}, \frac{393}{109}, \frac{649}{180}, \dots$$

and for each, we test whether $a^2 - 13b^2 = 1$. The first solution we get is $(649, 180)$.

Now, by the theorem, every solution (x_k, y_k) arises of the form

$$(x_k, y_k) = (649 + 180\sqrt{13})^k.$$

- (2) Closed formulas for solutions to Pell's equations.

¹As in class, in terms of coefficients powers of some $a + b\sqrt{D}$.

- (a) Explain why the k th positive solution (x_k, y_k) of the Pell's equation $x^2 - 2y^2 = 1$ satisfies the equation

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

- (b) Diagonalize the matrix $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and use this to give a closed expression for (x_k, y_k) in terms of k . Your formulas should be in terms of particular linear combinations of powers of two numbers.
- (c) Use² your formulas from the previous part to show that

$$x_k = \left\lfloor \frac{(3 + 2\sqrt{2})^k}{2} \right\rfloor \quad \text{and} \quad y_k = \left\lfloor \frac{(3 + 2\sqrt{2})^k}{2\sqrt{2}} \right\rfloor.$$

Use this to quickly write down the first seven positive solutions to the Pell's equation $x^2 - 2y^2 = 1$.

- (d) Repeat the steps above with the appropriate numbers for the Pell's equation $x^2 - 5y^2 = 1$.

- (a) We know that $x_{k+1} + y_{k+1}\sqrt{2} = (x_k + y_k\sqrt{2})(3 + 2\sqrt{2})$, so $\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}$.

We also have $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. It then follows from a straightforward induction.

- (b) Using linear algebra, we obtain

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 + 2\sqrt{2} & 0 \\ 0 & 3 - 2\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix},$$

so

$$\begin{aligned} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (3 + 2\sqrt{2})^k & 0 \\ 0 & (3 - 2\sqrt{2})^k \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(3+2\sqrt{2})^k + (3-2\sqrt{2})^k}{2} \\ \frac{(3+2\sqrt{2})^k - (3-2\sqrt{2})^k}{2\sqrt{2}} \end{bmatrix}. \end{aligned}$$

- (c) Since $0 < 3 - 2\sqrt{2} < 1$, we have $0 < (3 - 2\sqrt{2})^k < 1$. Then, as $x_k = \frac{(3+2\sqrt{2})^k + (3-2\sqrt{2})^k}{2}$ is an integer, and $\frac{(3-2\sqrt{2})^k}{2} = x_k - \frac{(3+2\sqrt{2})^k}{2}$ is between 0 and 1, we must have that x_k is the round up of $\frac{(3+2\sqrt{2})^k}{2}$. Similar considerations yield the formula for y_k . We can just plug into a dollar store calculator to get solutions: for example, $\frac{(3+2\sqrt{2})^7}{2} = 114242.999998$ and $\frac{(3+2\sqrt{2})^7}{2\sqrt{2}} = 80782.0000015$, so $(x_7, y_7) = (114243, 80782)$.

- (d) We follow the same steps, now diagonalizing $\begin{bmatrix} 9 & 20 \\ 4 & 9 \end{bmatrix}$ and multiplying by $(1, 0)$ to get $x_k = \frac{(9+4\sqrt{5})^k - (9-4\sqrt{5})^k}{2}$, $y_k = \frac{(9+4\sqrt{5})^k - (9-4\sqrt{5})^k}{2\sqrt{5}}$. Again $(9 - 4\sqrt{5})^k < 1$ makes the rounding work. Then $\frac{(9+4\sqrt{5})^7}{2} = 299537288.999999999$ and $\frac{(9+4\sqrt{5})^7}{2\sqrt{5}} = 133957148.00000000037$, so $(x_7, y_7) = (299537289, 133957148)$.

- (3) Not solving $x^2 - Dy^2 = -1$: Let $D > 1$ be a positive integer that is not a perfect square.

- (a) Show that if $D \equiv 0 \pmod{4}$ or $D \equiv 3 \pmod{4}$, then the equation $x^2 - Dy^2 = -1$ has no integer solutions.

²Recall that $\lfloor x \rfloor$ denotes the greatest integer n such that $n \leq x$ and $\lceil x \rceil$ denotes the smallest integer n such that $n \geq x$.

- (b) Show that if $q \equiv 3 \pmod{4}$ is prime and $q \mid D$, then the equation $x^2 - Dy^2 = -1$ has no integer solutions.

- (a) We know that $x^2, y^2 \equiv 0, 1 \pmod{4}$. If $D \equiv 0 \pmod{4}$, then $x^2 - Dy^2$ is congruent to 0 or 1 modulo 4, whereas -1 is congruent to 3 mod 4 so there can be no solutions. If $D \equiv 3 \pmod{4}$, then $x^2 - Dy^2 \equiv x^2 + y^2 \pmod{4}$, and the possible values are 0, 1, 2 (mod 4), so there again can be no solution.
- (b) Suppose that $q \equiv 3 \pmod{4}$ and $q \mid D$. Given a solution $x^2 - Dy^2 = -1$, we obtain $x^2 \equiv -1 \pmod{q}$. By QR part -1 , this has no solutions since -1 is not a quadratic residue in this case. Thus, $x^2 - Dy^2 = -1$ has no solutions.

- (4) Solving $x^2 - Dy^2 = -1$: Let $D > 1$ be a positive integer that is not a perfect square.
- (a) Show that if (c, d) is a positive integer solution to $x^2 - Dy^2 = -1$, then $\frac{c}{d}$ is a convergent in the continued fraction expansion of \sqrt{D} .
- (b) Show that if (c, d) is a positive integer solution to $x^2 - Dy^2 = -1$, (a, b) is a positive integer solution to $x^2 - Dy^2 = 1$, and

$$e + f\sqrt{D} = (a + b\sqrt{D})(c + d\sqrt{D}),$$

then (e, f) is another positive integer solution to $x^2 - Dy^2 = -1$.

- (c) Describe infinitely many solutions to the equation $x^2 - 13y^2 = -1$.

- (a) We have that $1 = |c^2 - d^2D| = |c + d\sqrt{D}| \cdot |c - d\sqrt{D}|$, so $\left| \frac{c}{d} + \sqrt{D} \right| \cdot \left| \frac{c}{d} - \sqrt{D} \right| = \frac{1}{d^2}$. Since $\left| \frac{c}{d} - \sqrt{D} \right| \leq 1$ and $\sqrt{D} > 1$, we have $\left| \frac{c}{d} + \sqrt{D} \right| > 2$, so $\left| \frac{c}{d} - \sqrt{D} \right| < \frac{1}{2d^2}$. By the Theorem on Good Approximations and convergents, this implies that $\frac{c}{d}$ is a convergent.
- (b) We have $N(e + f\sqrt{D}) = N(a + b\sqrt{D})N(c + d\sqrt{D})$. Since (a, b) is a solution to Pell's equation, $N(a + b\sqrt{D}) = 1$, and the given equation implies that $N(c + d\sqrt{D}) = -1$, so $N(e + f\sqrt{D}) = -1$, which means that (e, f) is a solution to the given equation.
- (c) From the convergent $\frac{18}{5}$ computed above, we get the first solution $18^2 - 13 \cdot 5^2 = -1$. Then any (x_k, y_k) such that $x_k + y_k\sqrt{D} = (649 + 180\sqrt{13})^k (18 + 5\sqrt{5})$ for $k \geq 0$ is a solution of the given equation.

The remaining problem is only required for Math 845 students, though all are encouraged to think about it.

- (5) Let D be a positive integer that is not a perfect square. Suppose that $x^2 - Dy^2 = -1$ has a solution, and let (c, d) be the smallest positive integer solution. Let (a, b) be the smallest integer solution to the Pell's equation $x^2 - Dy^2 = 1$. Show that $(c + d\sqrt{D})^2 = a + b\sqrt{D}$, and use this to describe all solutions to $x^2 - Dy^2 = -1$ in terms of c and d .

Let (c, d) be the smallest positive integer solution to $x^2 - Dy^2 = -1$ and (a, b) be the smallest integer solution to the Pell's equation $x^2 - Dy^2 = 1$. Note first that $\alpha = a + b\sqrt{D}$ has $N(\alpha) = 1$ and $\gamma = c + d\sqrt{D}$ has $N(\gamma) = -1$. In particular, $N(\gamma^2) = 1$, so γ^2 is some positive solution to the Pell's equation $x^2 - Dy^2 = 1$. Based on our results on Pell's equation, we must have $\gamma^2 = \alpha^k$ for some $k \geq 1$.

We consider the elements of $\mathbb{Z}[\sqrt{D}]$:

$$\alpha\gamma^{-1} = (a + b\sqrt{D})(d\sqrt{D} - c) = (bd\sqrt{D} - ac) + (ad - bc)\sqrt{D}$$

$$\alpha^{-1}\gamma = (a - b\sqrt{D})(c + d\sqrt{D}) = (ac - bd\sqrt{D}) + (ad - bc)\sqrt{D}.$$

Note that $ad - bc > 0$ since $a > b\sqrt{D}$ and $d\sqrt{D} > c$, so $ad\sqrt{D} > bc\sqrt{D}$.

We claim that $bd\sqrt{D} - ac > 0$. To see this, first, if equality holds, then $-1 = N(\alpha\gamma^{-1}) = -(ad - bc)^2 D$ is a contradiction. If $bd\sqrt{D} - ac < 0$, then $N(\alpha^{-1}\gamma) = -1$ and the coefficients of $\alpha^{-1}\gamma$ yield a positive integer solution to $x^2 - Dy^2 = -1$; say (e, f) . But then $\gamma = (\alpha^{-1}\gamma)\alpha = (e + f\sqrt{D})(a + b\sqrt{D})$ is easily seen to have larger positive coefficients than (e, f) , which contradicts minimality of (c, d) . This establishes the claim.

Thus, the coefficients of $\alpha\gamma^{-1}$ yield a positive solution to $x^2 - Dy^2 = -1$; say (e, f) ; set $\varepsilon = e + f\sqrt{D}$. By an argument similar to above, we have (e, f) is less than (a, b) . Then, the coefficients of ε^2 are less than those of α^2 . Since $N(\varepsilon^2) = (-1)^2 = 1$, the coefficients of ε^2 are a solution to the Pell's equation $x^2 - Dy^2 = 1$, and since every positive solution comes from a power of α , we must have $\varepsilon^2 = \alpha$. That is: $(\alpha\gamma^{-1})^2 = \alpha$, so $\gamma^2 = \alpha$. This is what we wanted to show.