Math 445 — Problem Set #6
Due: Friday, November 3 by 7 pm, on Canvas

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times. Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Use the methods from class to give a formula[1] for all solutions of the Pell's equation

$$x^2 - 13y^2 = 1.$$

(2) Closed formulas for solutions to Pell's equations.

(a) Explain why the kth positive solution (x_k, y_k) of the Pell's equation $x^2 - 2y^2 = 1$ satisfies the equation

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(b) Diagonalize the matrix $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and use this to give a closed expression for (x_k, y_k) in terms of k. Your formulas should be in terms of particular linear combinations of powers of two numbers.

(c) Use[2] your formulas from the previous part to show that

$$x_k = \left\lceil \frac{(3+2\sqrt{2})^k}{2} \right\rceil \qquad \text{and} \qquad y_k = \left\lfloor \frac{(3+2\sqrt{2})^k}{2\sqrt{2}} \right\rfloor.$$

Use this to quickly write down the first seven positive solutions to the Pell's equation $x^2 - 2y^2 = 1$.

(d) Repeat the steps above with the appropriate numbers for the Pell's equation $x^2 - 5y^2 = 1$.

(3) Not solving $x^2 - Dy^2 = -1$: Let $D > 1$ be a positive integer that is not a perfect square.

(a) Show that if $D \equiv 0 \pmod 4$ or $D \equiv 3 \pmod 4$, then the equation $x^2 - Dy^2 = -1$ has no integer solutions.

(b) Show that if $q \equiv 3 \pmod 4$ is prime and $q \mid D$, then the equation $x^2 - Dy^2 = -1$ has no integer solutions.

(4) Solving $x^2 - Dy^2 = -1$: Let $D > 1$ be a positive integer that is not a perfect square.

(a) Show that if (c, d) is a positive integer solution to $x^2 - Dy^2 = -1$, then $\frac{e}{f}$ is a convergent in the continued fraction expansion of $\sqrt{D}$.

(b) Show that if (c, d) is a positive integer solution to $x^2 - Dy^2 = -1$, (a, b) is a positive integer solution to $x^2 - Dy^2 = 1$, and

$$e + f\sqrt{D} = (a + b\sqrt{D})(c + d\sqrt{D}),$$

then (e, f) is another positive integer solution to $x^2 - Dy^2 = -1$.

(c) Describe infinitely many solutions to the equation $x^2 - 13y^2 = -1$.

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[1] As in class, in terms of coefficients powers of some $a + b\sqrt{D}$.

[2] Recall that $\lfloor x \rfloor$ denotes the greatest integer n such that $n \leq x$ and $\lceil x \rceil$ denotes the smallest integer n such that $n \geq x$.

The remaining problem is only required for Math 845 students, though all are encouraged to think about it.

(5) Let D be a positive integer that is not a perfect square. Suppose that $x^2 - Dy^2 = -1$ has a solution, and let (c, d) be the smallest positive integer solution. Let (a, b) be the smallest integer solution to the Pell's equation $x^2 - Dy^2 = 1$. Show that $(c + d\sqrt{D})^2 = a + b\sqrt{D}$, and use this to describe all solutions to $x^2 - Dy^2 = -1$ in terms of c and d.