Math 445 — Problem Set #5
Due: Friday, October 20 by 7 pm, on Canvas

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times. Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) The continued fraction expansion of Euler's constant e is given by

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots].$$

Use this and results from class to find a rational approximation of e that is accurate to four digits (beyond the decimal place) without using any other knowledge about the number e.

Taking the convergent $C_8 = [2; 1, 2, 1, 1, 4, 1, 1, 6] = \frac{1264}{465}$, by Dirichlet's theorem, we have $|e - C_8| < \frac{1}{465^2} < \frac{1}{10^4}$.

(2) Find the real number with continued fraction expansion

$$[1; 2, 3, 2, 3, 2, 3, \dots] \quad \text{(and repeats forever like so).}$$

Write $x = [1; 2, 3, 2, 3, 2, 3, \dots]$. Then $x+2 = 3 + \frac{1}{2+\frac{1}{3+\frac{1}{2+\frac{1}{\ddots}}}}$. Thus, $x+2 = 3 + \frac{1}{2+\frac{1}{x+2}}$. Simplifying, we get $2x^2 + 2x - 7 = 0$, which yields $\frac{-1\pm 2\sqrt{15}}{4}$. Only one root is positive, so this must be the number with this expansion.

(3) Let $d \geq 2$ be a positive integer.

(a) Show that the continued fraction expansion of $\sqrt{d^2+1}$ is

$$\sqrt{d^2+1} = [d; 2d, 2d, 2d, 2d, 2d, 2d \dots] \quad \text{(and repeats forever like so).}$$

(b) Show that the continued fraction expansion of $\sqrt{d^2-1}$ is

$$\sqrt{d^2-1} = [d-1; 1, 2d-2, 1, 2d-2, 1, 2d-2, \dots] \quad \text{(and repeats forever like so).}$$

(c) Apply the previous parts to give continued fraction expansions for $\sqrt{101}$ and $\sqrt{63}$.

First, we start with $x = \sqrt{d^2+1}$. We run the continued fraction algorithm. Note that $\lfloor\sqrt{d^2+1}\rfloor = d$ so the first partial quotient is d. We then take

$$\frac{1}{\sqrt{d^2+1} - d} = \frac{\sqrt{d^2+1}+d}{\sqrt{d^2+1}^2 - d^2} = \sqrt{d^2+1} + d,$$

and repeat. Then

$$\lfloor\sqrt{d^2+1} + d\rfloor = 2d$$

Since $(\sqrt{d^2+1}+d) - 2d = \sqrt{d^2+1} - d$, the continued fraction algorithm continues in the same way as above; i.e., repeats in a loop. It follows that the continues fraction algorithm returns

$$[d; 2d, 2d, 2d, \dots]$$

Now, we consider $y = \sqrt{d^2-1}$. We run the continued fraction algorithm. Note that $\lfloor \sqrt{d^2-1} \rfloor = d-1$ since $(d-1)^2 < d^2-1 < d^2$ so the first partial quotient is $d-1$. We then take

$$\frac{1}{\sqrt{d^2-1}-(d-1)} = \frac{\sqrt{d^2-1}+(d-1)}{\sqrt{d^2-1}^2-(d-1)^2} = \frac{\sqrt{d^2-1}+(d-1)}{2d-2},$$

and repeat. We claim that this number is less than 2; indeed, since $d \geq 2$, we have

$$\begin{aligned} d+1 &< 9(d-1) \\ d^2-1 &< 9(d-1)^2 \\ \sqrt{d^2-1} &< 3(d-1) \\ \sqrt{d^2-1}+(d-1) &< 4(d-1) \\ \frac{\sqrt{d^2-1}+(d-1)}{2(d-1)} &< 2 \end{aligned}$$

so the next partial quotient must be 1. We continue the algorithm: we need to find the floor of

$$\begin{aligned} \left(\frac{\sqrt{d^2-1}+(d-1)}{2(d-1)} - 1\right)^{-1} &= \left(\frac{\sqrt{d^2-1}-(d-1)}{2(d-1)}\right)^{-1} \\ &= \frac{2(d-1)}{\sqrt{d^2-1}-(d-1)}\left(\frac{\sqrt{d^2-1}+(d-1)}{\sqrt{d^2-1}+(d-1)}\right) \\ &= \frac{2(d-1)(\sqrt{d^2-1}+(d-1))}{2(d-1)} = \sqrt{d^2-1}+(d-1). \end{aligned}$$

Since $\lfloor \sqrt{d^2-1} \rfloor = d-1$, we have $\lfloor \sqrt{d^2-1} \rfloor = 2(d-1)$. After subtracting the floor, we get $\sqrt{d^2-1}-(d-1)$, and the continued fraction algorithm returns to the same value as after the 0th partial quotient. Thus, the algorithm will repeat the same values from that point, namely $1, 2d-2$, and back again to $1, 2d-2$, and so on. We conclude that the continued fraction is

$$[d-1; 1, 2d-2, 1, 2d-2, \ldots].$$

(4) In this problem, we will prove the following theorem, which basically says that the convergents are the *best* approximations of a real number by a rational number.
THEOREM: Let r be a real number, $C_k = \frac{p_k}{q_k}$ be the k-th convergent of r, and $\frac{p}{q} \neq r$ be a rational number, with $q > 0$. If $q < q_k$, then $\left| r - \frac{p}{q} \right| > \left| r - \frac{p_k}{q_k} \right|$.
- (a) Set $u = (-1)^k(q_k p - p_k q)$ and $v = (-1)^k(p_{k+1} q - q_{k+1} p)$. Show that $p_{k+1}u + p_k v = p$ and $q_{k+1}u + q_k v = q$.
- (b) Show[1] that $u, v \neq 0$, and that[2] u and v have opposite signs.
- (c) Show that $q_k r - p_k$ and $q_{k+1} r - p_{k+1}$ have opposite signs.
- (d) Show that $|qr - p| = |u(q_{k+1} r - p_{k+1}) + v(q_k r - p_k)| \geq |q_k r - p_k|$ and conclude the proof.

[1] Hint: Use the Proposition from class to show that p_k, q_k are coprime, and use this to show that $u = 0$ implies $q_k | q$.
[2] Hint: Use the second equation from part (a).

(a) We plug in the values:

$$\begin{aligned} p_{k+1}u + p_k v &= (-1)^k p_{k+1}(q_k p - p_k q) + (-1)^k p_k(p_{k+1}q - q_{k+1}p) \\ &= (-1)^k p(p_{k+1}q_k - p_k q_{k+1}) = (-1)^k p(-1)^k = p, \end{aligned}$$

and

$$\begin{aligned} q_{k+1}u + q_k v &= (-1)^k q_{k+1}(q_k p - p_k q) + (-1)^k q_k(p_{k+1}q - q_{k+1}p) \\ &= (-1)^k q(p_{k+1}q_k - p_k q_{k+1}) = (-1)^k q(-1)^k = q, \end{aligned}$$

where Proposition 66 was used in the last step.

(b) By Proposition 66, we know that 1 can be realized as a linear combination of p_k and q_k, and hence $\gcd(p_k, q_k) = 1$. Suppose that $u = 0$. Then $q_k p = p_k q$ implies $q_k \mid p_k q$, which implies $q_k \mid q$, and in particular $q_k \leq q$, a contradiction. Now, if $v = 0$, then $q = q_{k+1}u$, so $q_k < q_{k+1} \leq q$. Finally, to see that u and v have opposite signs, note that if u, v are both negative, then $q_{k+1}u + q_k v$ is negative, contradicting that $q_{k+1}u + q_k v = q > 0$; if u, v are both positive, then $q_{k+1}u + q_k v > q$, a contradiction.

(c) From Worksheet #11 problem #8, we know that $r - \frac{p_\ell}{q_\ell} - r \geq 0$ if ℓ is even and $r - \frac{p_\ell}{q_\ell} \leq 0$ if ℓ is odd. Multiplying by q_ℓ, the signs of $q_\ell - rp_\ell$ follow the same rule. Since one of $k, k+1$ is even and the other odd, the claim follows.

(d) Using (a), we have

$$qr - p = (q_{k+1}u + q_k v)r - (p_{k+1}u + p_k v) = u(q_{k+1}r - p_{k+1}) + v(q_k r - p_k).$$

Since u, v have opposite signs and $(q_{k+1}r - p_{k+1})$, $(q_k r - p_k)$ have opposite signs, the two terms in the sum above, $u(q_{k+1}r - p_{k+1})$ and $v(q_k r - p_k)$ have the same sign. Thus,

$$|qr - p| = |u(q_{k+1}r - p_{k+1})| + |v(q_k r - p_k)| \geq |v(q_k r - p_k)| \geq |q_k r - p_k|.$$

Dividing through by q_k, we get

$$\left| r - \frac{p_k}{q_k} \right| \leq \frac{q}{q_k} \left| r - \frac{p}{q} \right| < \left| r - \frac{p}{q} \right|.$$