Math 445 — Problem Set #5
Due: Friday, October 20 by 7 pm, on Canvas

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times. Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) The continued fraction expansion of Euler's constant e is given by

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots].$$

Use this and results from class to find a rational approximation of e that is accurate to four digits (beyond the decimal place) without using any other knowledge about the number e.

(2) Find the real number with continued fraction expansion

$$[1; 2, 3, 2, 3, 2, 3, \dots] \quad \text{(and repeats forever like so).}$$

(3) Let $d \geq 2$ be a positive integer.

(a) Show that the continued fraction expansion of $\sqrt{d^2+1}$ is

$$\sqrt{d^2+1} = [d; 2d, 2d, 2d, 2d, 2d, 2d \dots] \quad \text{(and repeats forever like so).}$$

(b) Show that the continued fraction expansion of $\sqrt{d^2-1}$ is

$$\sqrt{d^2-1} = [d-1; 1, 2d-2, 1, 2d-2, 1, 2d-2, \dots] \quad \text{(and repeats forever like so).}$$

(c) Apply the previous parts to give continued fraction expansions for $\sqrt{101}$ and $\sqrt{63}$.

(4) In this problem, we will prove the following theorem, which basically says that the convergents are the *best* approximations of a real number by a rational number.

THEOREM: Let r be a real number, $C_k = \frac{p_k}{q_k}$ be the k-th convergent of r, and $\frac{p}{q} \neq r$ be a rational number, with $q > 0$. If $q < q_k$, then $\left| r - \dfrac{p}{q} \right| > \left| r - \dfrac{p_k}{q_k} \right|$.

(a) Set $u = (-1)^k(q_k p - p_k q)$ and $v = (-1)^k(p_{k+1} q - q_{k+1} p)$. Show that $p_{k+1}u + p_k v = p$ and $q_{k+1}u + q_k v = q$.

(b) Show[1] that $u, v \neq 0$, and that[2] u and v have opposite signs.

(c) Show that $q_k r - p_k$ and $q_{k+1} r - p_{k+1}$ have opposite signs.

(d) Show that $|qr - p| = |u(q_{k+1} r - p_{k+1}) + v(q_k r - p_k)| \geq |q_k r - p_k|$ and conclude the proof.

[1]Hint: Use the Proposition from class to show that p_k, q_k are coprime, and use this to show that $u = 0$ implies $q_k | q$.

[2]Hint: Use the second equation from part (a).